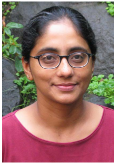
From Stable Sets to Sums of Squares and Conic Factorizations

Pablo A. Parrilo

Laboratory for Information and Decision Systems
Electrical Engineering and Computer Science
Massachusetts Institute of Technology



Based on joint work with **João Gouveia** and **Rekha Thomas** (U. Washington)

Outline

1. Stable set relaxations
   - The Stable Set Problem
   - Lovász's Theta Body
2. Theta Bodies of Ideals
   - Examples and Definitions
   - First Theta Body
3. Cone lifts of convex bodies
   - Conic extended formulations
   - Slack operators and cone ranks

The Problem

Our starting point is a classical problem in combinatorics:

Stable Set Problem

Given a graph $G = (V, E)$ and vertex weights ω find a stable set of vertices S for which the cost

$$\omega(S) := \sum_{s \in S} \omega_s$$

is maximum.

Remarks:

- If all weights are one, we are computing $\alpha(G)$, the cardinality of the largest independent set;
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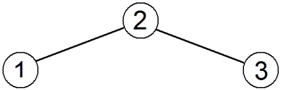
Stable Set Polytope

Given a graph $G = (\{1, ..., n\}, E)$ we define $\mathrm{STAB}(G)$, the **stable set polytope** of G, in the following way:

- For every stable set $S \subseteq \{1, \ldots, n\}$ consider its characteristic vector $\chi_S \in \{0, 1\}^n$;
- let $S_G \subset \{0, 1\}^n$ be the collection of all those vectors;
- the polytope $\mathrm{STAB}(G)$ is then defined as the convex hull of the vectors in S_G.

Example



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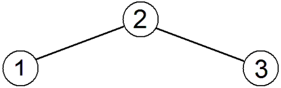

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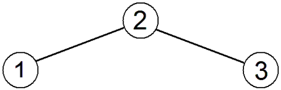

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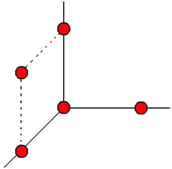

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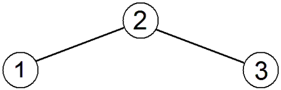

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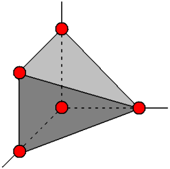

Reformulation

Stable Set Problem Reformulated

Given a graph $G = (\{1, ..., n\}, E)$ and a weight vector $\omega \in \mathbb{R}^n$, solve the linear program

$$\alpha(G, \omega) := \max_{x \in \mathrm{STAB}(G)} \langle \omega, x \rangle .$$

However, finding $\mathrm{STAB}(G)$ is as hard as solving the original problem, and not practical in general.

Want to find approximations for it.

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Definition of Theta Body

Definition (Lovász $\sim$ 1980)

Given a graph $G = (\{1, \ldots, n\}, E)$ we define its theta body, $\mathrm{TH}(G)$, as the set of all vectors $x \in \mathbb{R}^n$ such that

$$\begin{bmatrix} 1 & x^T \\ x & U \end{bmatrix} \succeq 0$$

for some symmetric $U \in \mathbb{R}^{n \times n}$ with $\mathrm{diag}(U) = x$ and $U_{ij} = 0$ for all $(i,j) \in E$.

- $\mathrm{STAB}(G) \subseteq \mathrm{TH}(G)$ since for all stable sets S,

$$0 \preceq (1, \chi_S) \cdot (1, \chi_S)^t = \begin{bmatrix} 1 & \chi_S^t \\ \chi_S & \chi_S \cdot \chi_S^t \end{bmatrix}.$$

Some Properties of the Theta Body

- Optimizing over the theta body is polynomial in the size of the graph.

Theorem (Lovász $\sim$ 1980)

The relaxation is tight, i.e. $TH(G) = STAB(G)$, if and only if the graph G is perfect.

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From combinatorics to algebra

Wonderful, and well-known.
Can we gain a better understanding, and generalize this?
Instead of characteristic vectors, let's think polynomials:

$$\begin{aligned} x_i \in \{0,1\} \quad &\Leftrightarrow \quad x_i(1-x_i) = 0 \\ \text{edge constraints} \quad &\Leftrightarrow \quad x_i x_j = 0 \quad (i,j) \in E. \end{aligned}$$

Why?

- Can use a simple algebraic proof system.
- Continuous and/or discrete variables.
- Same basic tools, independent of specific structure.
- Will be able to exploit additional features.
- Later, may want to go back to combinatorics.

Connection to Algebra

Let $I \subseteq \mathbb{R}[\mathbf{x}]$ be a polynomial ideal.

Definition

A polynomial $p(\mathbf{x})$ is **sos modulo the ideal** I if it can be written as a sum of squares of polynomials modulo I.

$$p(\mathbf{x}) = \sum_i q_i(\mathbf{x})^2 \mod I.$$

Definition

A polynomial $p(x)$ is k**-sos modulo the ideal** I if it can be written as a sum of squares of polynomials of degree at most k modulo I.

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SOS and SDP

We can *decide* if a polynomial is k-sos using SDP. Furthermore, we can *optimize* over the set of k-sos polynomials.

Remarks:

- Details important, but irrelevant for this talk.
- OK. Sketch: choose basis for quotient, write quadratic form, taking normal form yields linear equations.
- Here we assume we can compute normal forms over I.

Why abstract this out? Methods operate at the level of polynomials, not the matrices that represent them.

Stable sets as SOS

Theorem (Lovász $\sim$ 1993)

$TH(G) = STAB(G)$ if and only if any linear polynomial $f(\mathbf{x})$ that is non-negative on $STAB(G)$ is 1-sos modulo $\mathcal{I}(S_G)$.

This property does not depend on the graph, but only on the ideal $\mathcal{I}(S_G)$ and its variety.

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Perfect ideals

Lovász's Question

Which ideals are "perfect" i.e., for what ideals I is it true that any linear polynomial that is nonnegative in $\mathcal{V}_{\mathbb{R}}(I)$ is 1-sos modulo I?

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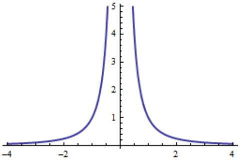
Example

Consider the ideal $I = \langle yx^2 - 1 \rangle$.



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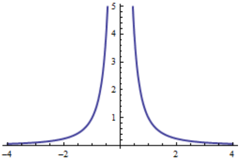

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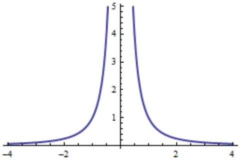

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Theta Bodies of Ideals

A geometric approach to the problem:

Definition

Given an ideal $I \subset \mathbb{R}[x_1, ..., x_n]$ we define its k-th theta body:

$$\mathrm{TH}_k(I) := \{\mathbf{p} \in \mathbb{R}^n : f(\mathbf{p}) \geq 0, \quad \forall \text{ linear } f \text{ that is } k\text{-sos mod } I\}.$$

Remarks:

- Nested closed convex sets:

$$\mathrm{TH}_1(I) \supseteq \mathrm{TH}_2(I) \supseteq \cdots \supseteq \overline{\mathrm{conv}(\mathcal{V}_{\mathbb{R}}(I))}.$$

- For any graph G, $\mathrm{TH}_1(\mathcal{I}(S_G)) = \mathrm{TH}(G)$.

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Finite convergence

Recall that a polynomial ideal is **radical** if $I = \mathcal{I}(\mathcal{V}(I))$ (informally, "no multiplicities").

Theorem (P.)

If I is a radical ideal whose variety is zero-dimensional then $TH_k(I) = conv(\mathcal{V}_{\mathbb{R}}(I))$ *for some k.*

Easy: existence of Lagrange interpolants, and weak Nullstellensatz.

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Theta Bodies and Nonnegativity

We call an ideal **TH_k-exact** if $\mathrm{TH}_k(I) = \overline{\mathrm{conv}(\mathcal{V}_{\mathbb{R}}(I))}$.

If I is k-sos, then clearly it is TH_k-exact. Under mild conditions, the converse is also true.

Theorem

Let I be a real radical ideal. Then I is k-sos if and only if it is TH_k-exact.

The real radical assumption cannot be dropped.

The ideal $I = \langle x^2 \rangle$ is not k-sos, but $\mathrm{TH}_1(I) = \overline{\mathrm{conv}(\mathcal{V}_{\mathbb{R}}(I))}$.

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Structural Result

We'll focus now on the first relaxation.

Theorem

Given any ideal $I \subseteq \mathbb{R}[\mathbf{x}]$ we have

$$TH_1(I) = \bigcap_{F \text{ convex quadric } \in I} conv(\mathcal{V}_{\mathbb{R}}(F)).$$

Consequences:

- If F is a convex quadric then $\langle F \rangle$ is TH_1-exact.
- There are arbitrarily high dimensional TH_1-exact ideals.

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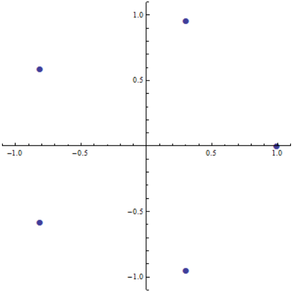

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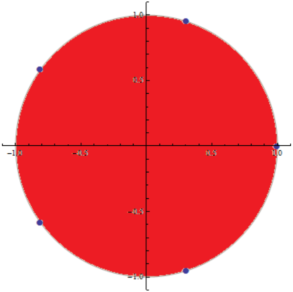

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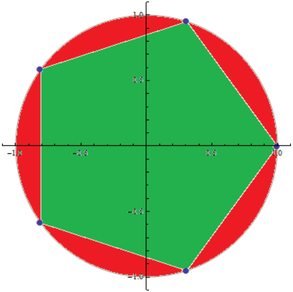

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Let S be the set $\{(0,0),(1,0),(0,1),(2,2)\}$. All convex quadrics that contain these four points are convex combinations of two particular parabolas.

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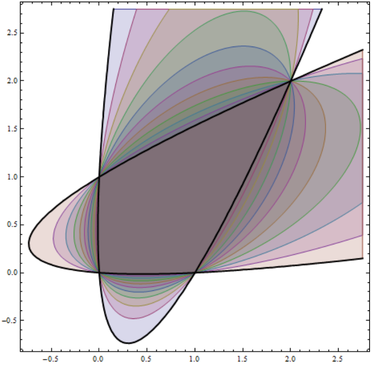

Zero-dimensional Varieties

A full characterization is possible in the case of zero-dimensional real radical ideals.

Theorem (Gouveia-P.-Thomas)

Let I be a zero-dimensional real radical ideal, then the following are equivalent:

- *I is 1-sos;*
- *I is TH_1-exact;*
- *For every facet defining hyperplane H of the polytope $conv(\mathcal{V}_{\mathbb{R}}(I))$ we have a parallel translate H' of H such that $\mathcal{V}_{\mathbb{R}}(I) \subseteq H' \cup H$.*
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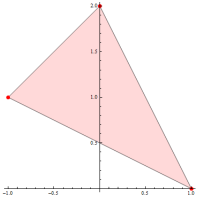
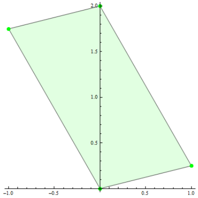
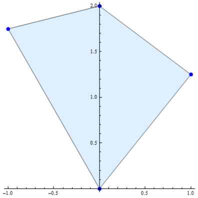
Examples in $\mathbb{R}^2$

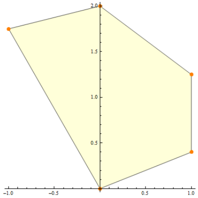


TH_1-exact



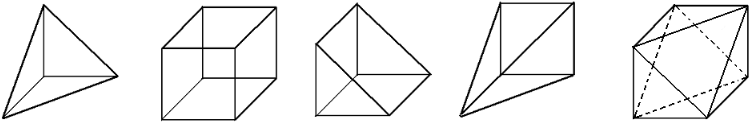


Not TH_1-exact



Examples in $\mathbb{R}^3$

TH_1-exact



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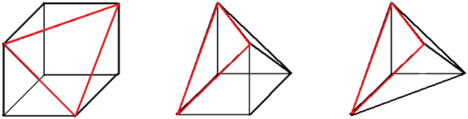

A Small Extension

Theorem

Suppose $S \subseteq \mathbb{R}^n$ is a finite point set such that for each facet F of $conv(S)$ there is an hyperplane H_F such that $H_F \cap conv(S) = F$ and S is contained in at most $t + 1$ parallel translates of H_F. Then $\mathcal{I}(S)$ is TH_t-exact.

Sufficient, but *not* necessary.

Consequences

Corollary

Let $S \subset \mathbb{R}^n$ be an exact set (i.e. with TH_1-exact vanishing ideal). Then

- *all points of S are vertices of $conv(S)$,*
- *the set of vertices of any face of $conv(S)$ is again exact,*
- *$conv(S)$ is affinely equivalent to a $0/1$ polytope.*

For simplicity, we'll call a finite set of points in $\mathbb{R}^n$ exact, if its vanishing ideal is TH_1-exact.

Theorem

If $S \subseteq \mathbb{R}^n$ is a finite exact point set then $\mathrm{conv}(S)$ has at most 2^d facets and vertices, where $d = \dim \mathrm{conv}(S)$. Both bounds are sharp.

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Perfect Graphs revisited

Corollary

A graph G is perfect if and only if for any facet supporting hyperplane H of its stable set polytope there is some hyperplane H' parallel to H such that $S_G \subseteq H \cup H'$.

Corollary

Let $P \subseteq \mathbb{R}^n$ be a full-dimensional down-closed $0/1$-polytope and S be its vertex set. Then S is exact if and only if P is the stable set polytope of a perfect graph.

Cone lifts of convex bodies

When does a convex body C have a "conic extended formulation"?

Definition

Let $K \subset \mathbb{R}^m$ be a closed convex cone and $C \subset \mathbb{R}^n$ a full-dimensional convex body. A *K-lift* of C is a set $Q = K \cap L$, where $L \subset \mathbb{R}^m$ is an affine subspace, and $\pi : \mathbb{R}^m \to \mathbb{R}^n$ is a linear map such that $C = \pi(Q)$.

If optimization over K is tractable, this is a "good" representation of C.

When do such representations exist?

Even ignoring complexity aspects, this question is not well understood. E.g: can all basic closed semialgebraic sets be represented using semidefinite programming?

Polars and slack operators

Recall that the *polar* of a convex set $C \subset \mathbb{R}^n$ is the set

$$C^\circ = \{y \in \mathbb{R}^n : \langle x, y \rangle \leq 1, \ \ \forall x \in C\}.$$

Let $\text{ext}(C)$ denote the set of *extreme points* of C.
Let $S : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be the operator defined by $S(x, y) = 1 - \langle x, y \rangle$.

Definition

The *slack operator* S_C of the convex set C is the restriction of S to $\text{ext}(C) \times \text{ext}(C^\circ)$.

When C is a polytope, then S_C is the usual slack matrix indexed by facets and vertices.

Cone factorizations and Generalized Yannakakis

Definition

The slack operator S_C is *K-factorizable* if there exist maps

$$A : \text{ext}(C) \to K \quad \text{and} \quad B : \text{ext}(C^\circ) \to K^*$$

such that $S_C(x, y) = \langle A(x), B(y) \rangle$ for all $(x, y) \in \text{ext}(C) \times \text{ext}(C^\circ)$.

Theorem (GPT 11)

If C has a proper K-lift then S_C is K-factorizable. Conversely, if S_C is K-factorizable then C has a K-lift.

Example

Let $C = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$. The set C has the semidefinite representation:

$$\begin{pmatrix} 1+x & y \\ y & 1-x \end{pmatrix} \succeq 0.$$

Thus, S_C must have a $\mathcal{S}^2_+$ factorization. Since $C^\circ = C$, we must have maps $A, B : S^2 \to \mathcal{S}^2_+$ such that for all $(x_1, y_1), (x_2, y_2) \in \text{ext}(C)$,

$$\langle A(x_1, y_1), B(x_2, y_2) \rangle = 1 - x_1 x_2 - y_1 y_2.$$

But this is accomplished by the maps

$$A(x_1, y_1) = \begin{pmatrix} 1+x_1 & y_1 \\ y_1 & 1-x_1 \end{pmatrix}, \qquad B(x_2, y_2) = \frac{1}{2} \begin{pmatrix} 1-x_2 & -y_2 \\ -y_2 & 1+x_2 \end{pmatrix}$$

which factorize S_C and can easily be checked to be psd.

Cone ranks

Assume we have a "nice" family of cones $\{K_k\}$ (e.g. $\{\mathbb{R}^k_+\}$ or $\{\mathcal{S}^k_+\}$).

Definition

The $\mathcal{K}$*-rank* of C, denoted by $\text{rank}_{\mathcal{K}}(C)$, is the least i for which $C = \pi(K_i \cap L)$ for some π and L.

Equivalently, this is asking for the least i for which the slack operator S_C has a $\mathcal{K}_i$-factorization.

Of particular interest are rank_+ and rank_{psd}, since they correspond to polyhedral or semidefinite lifts.

This makes sense

Some inequalities

- For any nonnegative matrix M

$$\frac{1}{2}\sqrt{1+8\,\mathrm{rank}(M)}-\frac{1}{2}\leq \mathrm{rank}_{psd}(M)\leq \mathrm{rank}_{+}(M).$$

- Gap between $\mathrm{rank}_+(M)$ and $\mathrm{rank}_{psd}(M)$ can be arbitrarily large:

$$M_{ij}=(i-j)^2=\left\langle \begin{pmatrix} i^2 & -i \\ -i & 1 \end{pmatrix}, \begin{pmatrix} 1 & j \\ j & j^2 \end{pmatrix} \right\rangle$$

has $\mathrm{rank}_{psd}(M)=2$, but $\mathrm{rank}_+(M)=\Omega(\log n)$.

Arbitrarily large gaps between all pairs of ranks (rank, rank_+ and $\mathrm{rank}_{\mathrm{psd}}$). For slack matrices of polytopes, arbitrarily large gaps between rank and rank_+, and rank and $\mathrm{rank}_{\mathrm{psd}}$.

Recently, Fiorini *et al.* established interesting links between rank_{psd} and quantum communication complexity, mirroring the situation between rank_+ and classical communication complexity.

Special case: 0-1 slacks

There is a simple, but important situation where rank(M) is an upper bound on $\text{rank}_{\text{psd}}(M)$.

Theorem

Take $M \in \mathbb{R}^{p\times q}$ and let M' be the nonnegative matrix obtained from M by squaring each entry of M. Then $\text{rank}_{psd}(M') \leq \text{rank}(M)$. In particular, if M is a $0/1$ matrix, $\text{rank}_{psd}(M) \leq \text{rank}(M)$.

Corollary

If a polytope in $\mathbb{R}^n$ has a $0/1$-slack matrix, then it admits a $\mathcal{S}_+^{n+1}$-lift. This follows since the rank of a slack matrix of a polytope in $\mathbb{R}^n$ is at most $n+1$.

A k-valued generalization is immediate.

Many questions

Conic factorizations and cone ranks are a good starting point to understand representability of convex sets. But much more work is needed!

- For polytopes, separations between rank_+ and rank_{psd} for slack matrices?
- Possible candidates: stable sets of perfect graphs?
- Algebraic obstructions?
- Approximate factorizations?
- Lower/upper bounds?

The End

Thank You!

Want to know more?

- J. Gouveia, P.A. Parrilo, and R. Thomas, Theta bodies for polynomial ideals, *SIAM J. Optim.*, Vol. 20, Issue 4, pp. 2097-2118, 2010.
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