Semidefinite programming and convex algebraic geometry

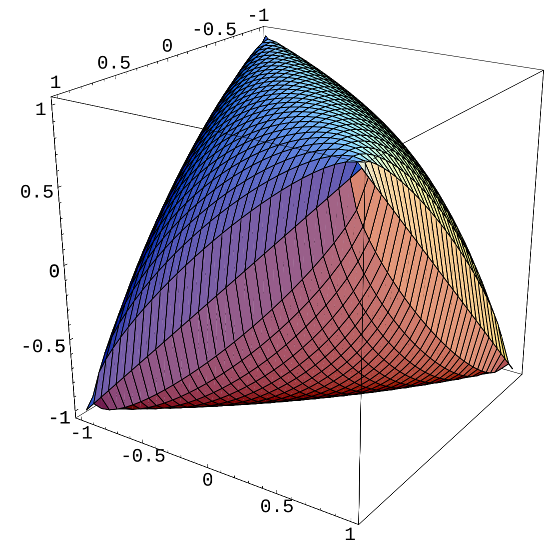
Pablo A. Parrilo

www.mit.edu/~parrilo

Laboratory for Information and Decision Systems
Electrical Engineering and Computer Science
Massachusetts Institute of Technology

This talk

- Convex sets with algebraic descriptions
- The role of semidefinite programming and sums of squares
- Unifying idea: convex hull of algebraic varieties
- Examples and applications throughout
- Discuss results, but also open questions
- Connections with other areas of mathematics



Convex sets: geometry vs. algebra

The geometric theory of convex sets (e.g., Minkowski, Carathéodory, Fenchel) is very rich and well-understood.

Enormous importance in applied mathematics and engineering, in particular in optimization.

But, what if we are concerned with the *representation* of these geometric objects? For instance, basic semialgebraic sets?

How do the *algebraic*, *geometric*, and *computational* aspects interact?

The polyhedral case

Consider first the case of *polyhedra*, which are described by finitely many *linear* inequalities $\{x \in \mathbb{R}^n : a_i^T x \leq b_i\}$.

- Behave well under projections (Fourier-Motzkin)
- Farkas lemma (or duality) gives emptiness certificates
- Good associated computational techniques
- Optimization over polyhedra is linear programming (LP)

The polyhedral case

Consider first the case of *polyhedra*, which are described by finitely many *linear* inequalities $\{x \in \mathbb{R}^n : a_i^T x \leq b_i\}$.

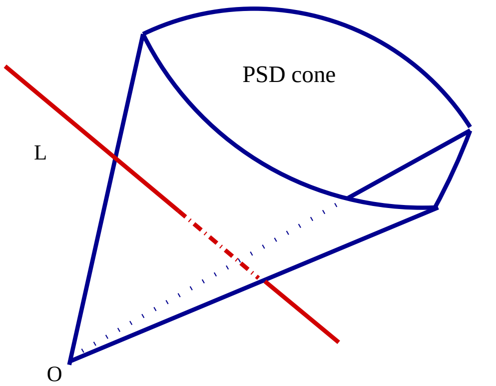
- Behave well under projections (Fourier-Motzkin)
- Farkas lemma (or duality) gives emptiness certificates
- Good associated computational techniques
- Optimization over polyhedra is linear programming (LP)

Great. But how to move away from linearity? For instance, if we want convex sets described by polynomial inequalities?

Claim: semidefinite programming is an essential tool.

Semidefinite programming (SDP, LMIs)

A broad generalization of LP to symmetric matrices



$$\min \operatorname{Tr} CX \qquad \text{s.t.} \quad X \in \mathcal{L} \cap \mathcal{S}^n_+$$

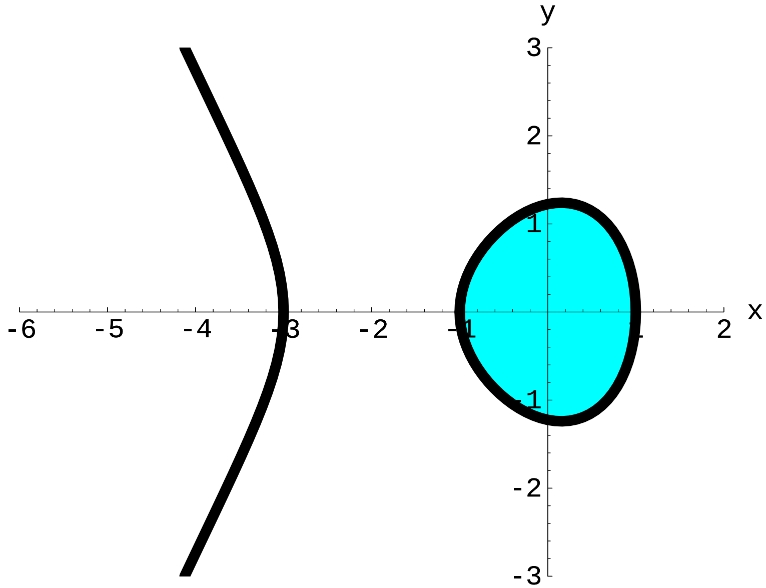
- Intersection of an affine subspace $\mathcal{L}$ and the cone of positive semidefinite matrices.
- *Lots* of applications. A true "revolution" in computational methods for engineering applications
- Originated in control theory and combinatorial optimization. Nowadays, applied everywhere.
- Convex finite dimensional optimization. Nice duality theory.
- Essentially, solvable in polynomial time (interior point, etc.)

Massachusetts Institute of Technology

Example

Consider the feasible set of the SDP:

$$\begin{bmatrix} x & 0 & y \\ 0 & 1 & -x \\ y & -x & 1 \end{bmatrix} \succeq 0.$$



- Convex, but not necessarily polyhedral
- In general, piecewise-smooth
- Determinant vanishes on the boundary

In this case, the determinant is the elliptic curve $x - x^3 = y^2$.

Symbolic vs. numerical computation

An ongoing discussion. Clearly, both have advantages/disadvantages.

- "Exact solutions" vs. "approximations"
- "Input data often inexact"
- "Global" vs. "local". One vs. all solutions.
- Computational models: bits vs. reals. Encoding of solutions.

"Best" method depends on the context. Hybrid symbolic-numeric methods are an interesting possibility.

SDP bring some interesting new twists.

Massachusetts Institute of Technology

Algebraic aspects of SDP

In LPs with rational data, the optimal solution is rational. Not so for SDP.

- Optimal solutions of relatively small SDPs generically have minimum defining polynomials of very high degree.
- Example (von Bothmer and Ranestad): For $n = 20$, $m = 105$, the algebraic degree of the optimal solution is $\approx 1.67 \times 10^{41}$.
- Explicit algebraic representations are absolutely impossible to compute (even without worrying about coefficient size!).
- Nevertheless, interior point methods yield arbitrary precision numerical approximations!

SDP provides an efficient, and numerically convenient *encoding*. Representation does not pay the price of high algebraic complexity.

For more about the algebraic degree of SDP, see Nie-Ranestad-Sturmfels (arXiv:math/0611562) and von Bothmer-Ranestad (arXiv:math.AG/0701877).

Semidefinite representations

A natural question in convex optimization:

What sets can be represented using semidefinite programming?

Semidefinite representations

A natural question in convex optimization:

What sets can be represented using semidefinite programming?

In the LP case, well-understood question: finite number of extreme points (polyhedral sets)

Semidefinite representations

A natural question in convex optimization:

What sets can be represented using semidefinite programming?

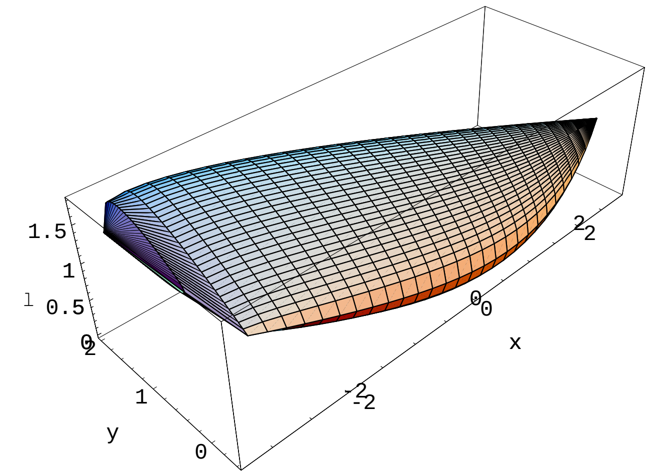
In the LP case, well-understood question: finite number of extreme points (polyhedral sets)

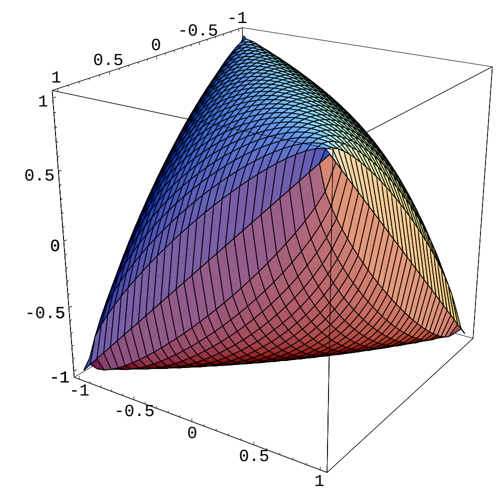
Are there "obstructions" to SDP representability?

Known SDP-representable sets

- Many interesting sets are known to be SDP-representable (e.g., polyhedra, convex quadratics, matrix norms, etc.)
- Preserved by "natural" properties: affine transformations, convex hull, polarity, etc.
- Several known structural results (e.g., facial exposedness)

Work of Nesterov-Nemirovski, Ramana, Tunçel, Güler, Renegar, Chua, etc.





Existing results

Obvious necessary conditions: $\mathcal{S}$ must be convex and semialgebraic.

Existing results

Obvious necessary conditions: $\mathcal{S}$ must be convex and semialgebraic.

Several versions of the problem:

- *Exact* vs. *approximate* representations.
- "Direct" (non-lifted) representations: no additional variables.

$$x \in \mathcal{S} \quad \Leftrightarrow \quad A_0 + \sum_i x_i A_i \succeq 0$$

- "Lifted" representations: can use extra variables (projection)

$$x \in \mathcal{S} \quad \Leftrightarrow \quad \exists y \text{ s.t. } A_0 + \sum_i x_i A_i + \sum y_j B_j \succeq 0$$

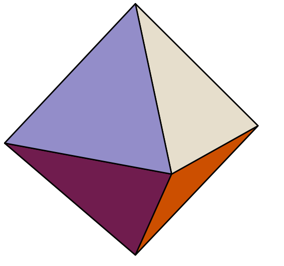
Projection helps a lot!

Liftings and projections

Often, "simpler" descriptions of convex sets from higher-dimensional spaces.

Ex: The n-dimensional crosspolytope (ℓ_1 unit ball). Requires 2^n linear inequalities, of the form

$$\pm x_1 \pm x_2 \pm \cdots \pm x_n \leq 1.$$



However, can efficiently represent it as the *projection* (on x) of:

$$\{(x, y) \in \mathbb{R}^{2n}, \quad \sum_{i=1}^{n} y_i \leq 1, \quad -y_i \leq x_i \leq y_i \quad i = 1, \ldots, n\}$$

Only $2n$ variables, and $2n + 1$ constraints!

In convexity, elimination is *not* always a good idea.

Quite the opposite, it is often advantageous to go to higher-dimensional spaces, where descriptions (can) become simpler.

Massachusetts Institute of Technology

Exact representations: direct case

$$x \in \mathcal{S} \quad \Leftrightarrow \quad A_0 + \sum_i x_i A_i \succeq 0$$

Exact representations: direct case

$$x \in \mathcal{S} \quad \Leftrightarrow \quad A_0 + \sum_i x_i A_i \succeq 0$$

Necessary condition: "rigid convexity." Every line through the set must intersect the Zariski closure of the boundary a constant number of times (equal to the degree of the curve).

[Assume $A_0 \succ 0$, and let $x_i = t\beta_i$. Then
$q(t) := \det(A_0 + \sum x_i A_i) = \det(A_0 + t \cdot \sum \beta_i A_i)$ has all its d roots real.]

Massachusetts Institute of Technology

Exact representations: direct case

$$x \in \mathcal{S} \quad \Leftrightarrow \quad A_0 + \sum_i x_i A_i \succeq 0$$

Necessary condition: "rigid convexity." Every line through the set must intersect the Zariski closure of the boundary a constant number of times (equal to the degree of the curve).

[Assume $A_0 \succ 0$, and let $x_i = t\beta_i$. Then
$q(t) := \det(A_0 + \sum x_i A_i) = \det(A_0 + t \cdot \sum \beta_i A_i)$ has all its d roots real.]

Helton & Vinnikov (2004) proved that in $\mathbb{R}^2$, this is also sufficient.

Related to hyperbolic polynomials and the Lax conjecture (Güler, Renegar, Lewis-P.-Ramana 2005)

For higher dimensions the problem is open.

Representations of hyperbolic polynomials

A homogeneous polynomial $p(x) \in \mathbb{R}[x_1, \ldots, x_n]$ is *hyperbolic* with respect to the direction $e \in \mathbb{R}^n$ if $t \mapsto p(x - te)$ has only real roots for all $x \in \mathbb{R}^n$.

Representations of hyperbolic polynomials

A homogeneous polynomial $p(x) \in \mathbb{R}[x_1, \ldots, x_n]$ is *hyperbolic* with respect to the direction $e \in \mathbb{R}^n$ if $t \mapsto p(x - te)$ has only real roots for all $x \in \mathbb{R}^n$.

Ex: Let A, B, C be symmetric matrices, with $A \succ 0$. The polynomial

$$p(x, y, z) = \det(Ax + By + Cz)$$

is hyperbolic wrt $e = (1, 0, 0)$ (eigenvalues of symm. matrices are real).

Massachusetts
Institute of
Technology

Representations of hyperbolic polynomials

A homogeneous polynomial $p(x) \in \mathbb{R}[x_1, \ldots, x_n]$ is *hyperbolic* with respect to the direction $e \in \mathbb{R}^n$ if $t \mapsto p(x - te)$ has only real roots for all $x \in \mathbb{R}^n$.

Ex: Let A, B, C be symmetric matrices, with $A \succ 0$. The polynomial

$$p(x, y, z) = \det(Ax + By + Cz)$$

is hyperbolic wrt $e = (1, 0, 0)$ (eigenvalues of symm. matrices are real).

Thm ("Lax Conjecture"): If $p(x, y, z)$ is hyperbolic wrt e, then it has such a determinantal representation.

Massachusetts Institute of Technology

Exact representations: lifted case

$$x \in \mathcal{S} \quad \Leftrightarrow \quad \exists y \text{ s.t. } A_0 + \sum_i x_i A_i + \sum y_j B_j \succeq 0$$

Exact representations: lifted case

$$x \in \mathcal{S} \quad \Leftrightarrow \quad \exists y \text{ s.t. } A_0 + \sum_i x_i A_i + \sum y_j B_j \succeq 0$$

No known nontrivial obstructions.

Does *every* convex basic SA set have a (lifted) exact SDP representation?

Exact representations: lifted case

$$x \in \mathcal{S} \quad \Leftrightarrow \quad \exists y \text{ s.t. } A_0 + \sum_i x_i A_i + \sum y_j B_j \succeq 0$$

No known nontrivial obstructions.

Does *every* convex basic SA set have a (lifted) exact SDP representation?

A recent positive partial answer.

Theorem (Helton & Nie 2007): Under strict positive curvature assumptions on the boundary, lifted SDP representations exist.

No direct control on lifting dimension.

For details, see `arXiv:0709.4017`

Exact representations: lifted case

$$x \in \mathcal{S} \quad \Leftrightarrow \quad \exists y \text{ s.t. } A_0 + \sum_i x_i A_i + \sum y_j B_j \succeq 0$$

No known nontrivial obstructions.

Does *every* convex basic SA set have a (lifted) exact SDP representation?

A recent positive partial answer.

Theorem (Helton & Nie 2007): Under strict positive curvature assumptions on the boundary, lifted SDP representations exist.

No direct control on lifting dimension.

For details, see `arXiv:0709.4017`

How are these representations obtained? Is this constructive at all?

SOS background

A multivariate polynomial $p(x)$ is a sum of squares (SOS) if

$$p(x) = \sum_i q_i^2(x), \quad q_i(x) \in \mathbb{R}[x].$$

- If $p(x)$ is SOS, then clearly $p(x) \geq 0 \; \forall x \in \mathbb{R}^n$.
- Converse not true, in general (Hilbert). Counterexamples exist.
- For univariate or quadratics, nongativity is equivalent to SOS.
- Convex condition, can be reduced to SDP.

Checking the SOS condition

Basic method, the "Gram matrix" (Shor 87, Choi-Lam-Reznick 95, Powers-Wörmann 98, Nesterov, Lasserre, P., etc.)

$F(x)$ is SOS iff $F(x) = w(x)^T Q w(x)$, where $w(x)$ is a vector of monomials, and $Q \succeq 0$.

Let $F(x) = \sum f_\alpha x^\alpha$. Index rows and columns of Q by monomials. Then,

$$F(x) = w(x)^T Q w(x) \qquad \Leftrightarrow \qquad f_\alpha = \sum_{\beta+\gamma=\alpha} Q_{\beta\gamma}$$

Thus, we have the SDP feasibility problem

$$f_\alpha = \sum_{\beta+\gamma=\alpha} Q_{\beta\gamma}, \qquad Q \succeq 0$$

- Factorize $Q = L^T L$. The SOS is given by $f = Lz$.

SOS Example

$$\begin{aligned} F(x,y) &= 2x^4 + 5y^4 - x^2y^2 + 2x^3y \\ &= \begin{bmatrix} x^2 \\ y^2 \\ xy \end{bmatrix}^T \begin{bmatrix} q_{11} & q_{12} & q_{13} \\ q_{12} & q_{22} & q_{23} \\ q_{13} & q_{23} & q_{33} \end{bmatrix} \begin{bmatrix} x^2 \\ y^2 \\ xy \end{bmatrix} \\ &= q_{11}x^4 + q_{22}y^4 + (q_{33} + 2q_{12})x^2y^2 + 2q_{13}x^3y + 2q_{23}xy^3 \end{aligned}$$

An SDP with equality constraints. Solving, we obtain:

$$Q = \begin{bmatrix} 2 & -3 & 1 \\ -3 & 5 & 0 \\ 1 & 0 & 5 \end{bmatrix} = L^T L, \qquad L = \frac{1}{\sqrt{2}} \begin{bmatrix} 2 & -3 & 1 \\ 0 & 1 & 3 \end{bmatrix}$$

And therefore $F(x,y) = \frac{1}{2}(2x^2 - 3y^2 + xy)^2 + \frac{1}{2}(y^2 + 3xy)^2$

Polynomial systems over $\mathbb{R}$

- When do equations and inequalities have real solutions?
- A remarkable answer: the Positivstellensatz.
- Centerpiece of real algebraic geometry (Stengle 1974).
- Common generalization of Hilbert's Nullstellensatz and LP duality.
- Guarantees the existence of algebraic infeasibility certificates for real solutions of systems of polynomial equations.
- Sums of squares are a fundamental ingredient.

How does it work?

P-satz and SOS

Given $\{x \in \mathbb{R}^n \mid f_i(x) \geq 0, \quad h_i(x) = 0\}$, want to *prove* that it is empty.

Define:

$$\text{Cone}(f_i) = \sum s_i \cdot (\prod_j f_j), \qquad \text{Ideal}(h_i) = \sum t_i \cdot h_i,$$

where the $s_i, t_i \in \mathbb{R}[x]$ and the s_i are sums of squares.

To prove infeasibility, find $f \in \text{Cone}(f_i), h \in \text{Ideal}(h_i)$ such that

$$f + h = -1.$$

- Can find certificates by solving SOS programs!
- Complete SOS hierarchy, by certificate degree (P. 2000).
- Directly provides hierarchies of bounds for optimization.

Massachusetts Institute of Technology

Convex hulls of algebraic varieties

Back to SDP representations...

Convex hulls of algebraic varieties

Back to SDP representations...

Focus here on a specific, but very important case.

Convex hulls of algebraic varieties

Back to SDP representations...

Focus here on a specific, but very important case.

Given a set $S \subset \mathbb{R}^n$, we can define its *convex hull*

$$\mathrm{conv} S := \left\{ \sum_i \lambda_i x_i \; : \; x_i \in S, \sum_i \lambda_i = 1, \; \lambda_i \geq 0 \right\}$$

We are interested in the case where S is a real algebraic variety.

Why?

Many interesting problems require or boil down *exactly* to understanding and describing convex hulls of (toric) algebraic varieties.

- Nonnegative polynomials and optimization
- Polynomial games
- Convex relaxations for minimum-rank

We discuss these next.

Polynomial optimization

Consider the unconstrained minimization of a multivariate polynomial

$$p(x) = \sum_{\alpha \in S} p_\alpha x^\alpha,$$

where $x \in \mathbb{R}^n$ and S is a given set of monomials (e.g., all monomials of total degree less than or equal to $2d$, in the dense case).

Define the (real, toric) algebraic variety $V_S \subset \mathbb{R}^{|S|}$:

$$V_S := \{(x^{\alpha_1}, \ldots, x^{\alpha_{|S|}}) \, : \, x \in \mathbb{R}^n\}.$$

This is the image of $\mathbb{R}^n$ under the monomial map (e.g., in the homogeneous case, the Veronese embedding).

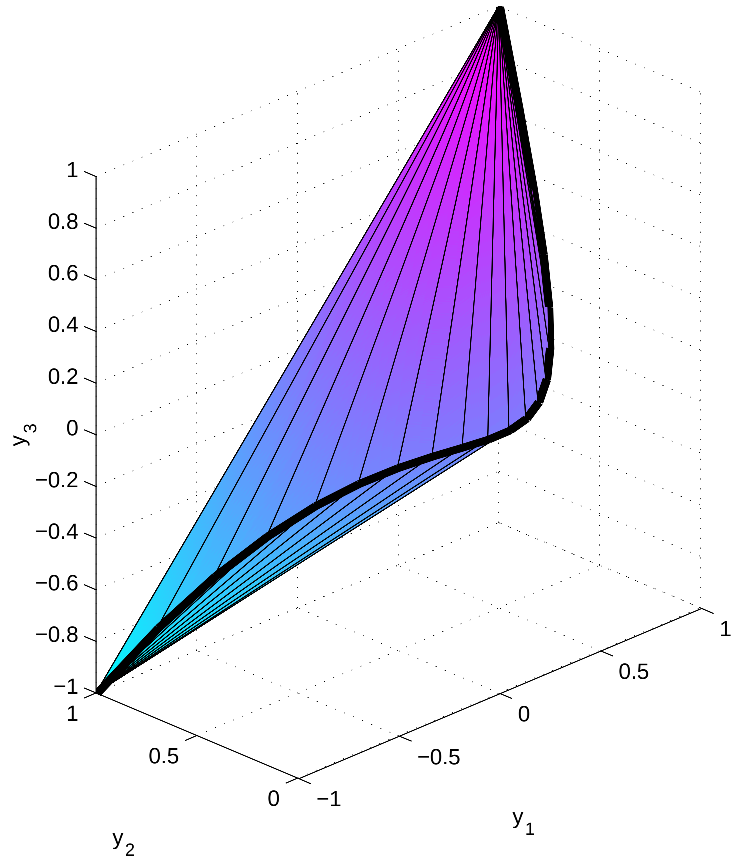
Want to study the *convex hull* of V_S. Extends to the constrained case.

Univariate case

Convex hull of the rational normal curve $(1, t, \ldots, t^d)$.

Not polyhedral.

Known geometry (Karlin-Shapley)



“Simplicial”: every supporting hyperplane yields a simplex.

Related to cyclic polytopes.

Polynomial optimization

We have then (almost trivially):

$$\inf_{x \in \mathbb{R}^n} p(x) = \inf\{p^T y \,:\, y \in \operatorname{conv} V_S\}$$

Optimizing a nonconvex polynomial is equivalent to linear optimization over a convex set (!)

Polynomial optimization

We have then (almost trivially):

$$\inf_{x\in\mathbb{R}^n} p(x) = \inf\{p^T y \; : \; y \in \operatorname{conv} V_S\}$$

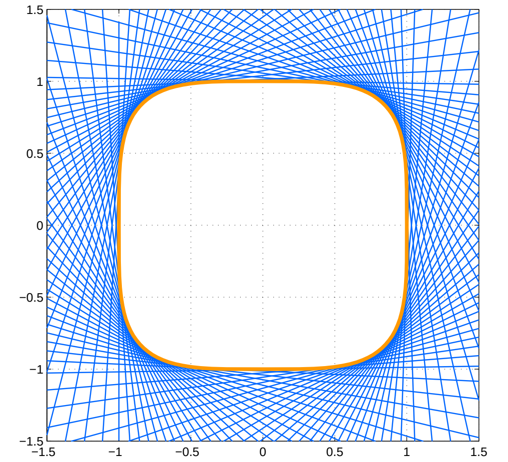
Optimizing a nonconvex polynomial is equivalent to linear optimization over a convex set (!)

Unfortunately, in general, it is NP-hard to check membership in $\operatorname{conv} V_S$.

Nevertheless, we can turn this around, and use SOS relaxations to obtain "good" approximate SDP descriptions of the convex hull V_S.

A "polar" viewpoint

Any convex set $\mathcal{S}$ is uniquely defined by its supporting hyperplanes.



Thus, if we can optimize a *linear function* over a set using SDP, we effectively have an SDP representation.

Need to solve (or approximate)

$$\min c^T x \qquad \text{s.t. } x \in \mathcal{S}$$

If $\mathcal{S}$ is defined by polynomial equations/inequalities, can use SOS.

Massachusetts Institute of Technology

A natural SOS approach

Let $\mathcal{S} = \{x \in \mathbb{R}^n \mid f_i(x) \geq 0\}$. Different conditions exist to certify nonnegativity of $c^T x + d$ over $\mathcal{S}$:

- General Positivstellensatz type:

$$(1+q)(c^T x + d) \in \textbf{cone}_{k+1}(f_i), \qquad q \in \textbf{cone}_k(f_i).$$

- Schmüdgen, Putinar/Lasserre:

$$c^T x + d \in \textbf{cone}_k(f_i), \quad \text{or} \quad c^T x + d \in \textbf{preprime}_k(f_i)$$

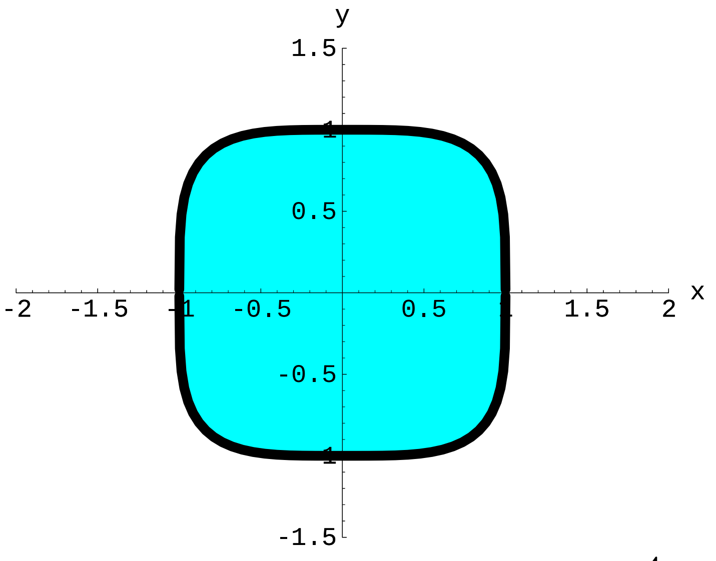
where $\textbf{preprime}_k \subseteq \textbf{cone}_k \subseteq \mathbb{R}_k[x]$. All these versions give convergent families of SDP approximations.

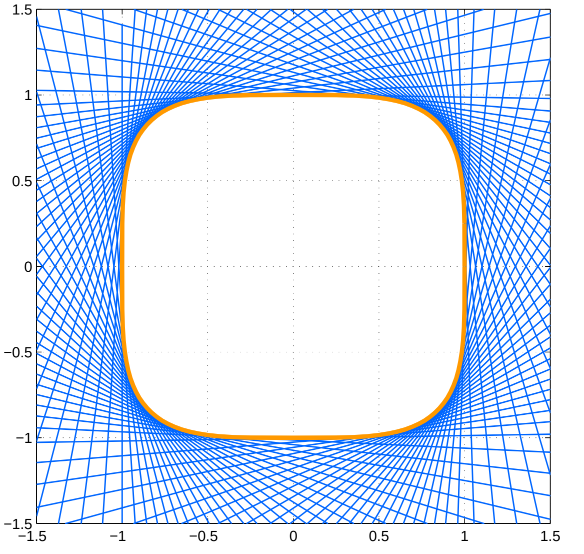
For instance, Putinar/Lasserre representations:

$$c^T x + d = s_0(x) + \sum_i s_i(x) f_i(x), \qquad s_0, s_i \textbf{ are SOS}.$$

An SDP representation of $\mathcal{S}$ exists if degree is *uniform*.

Example

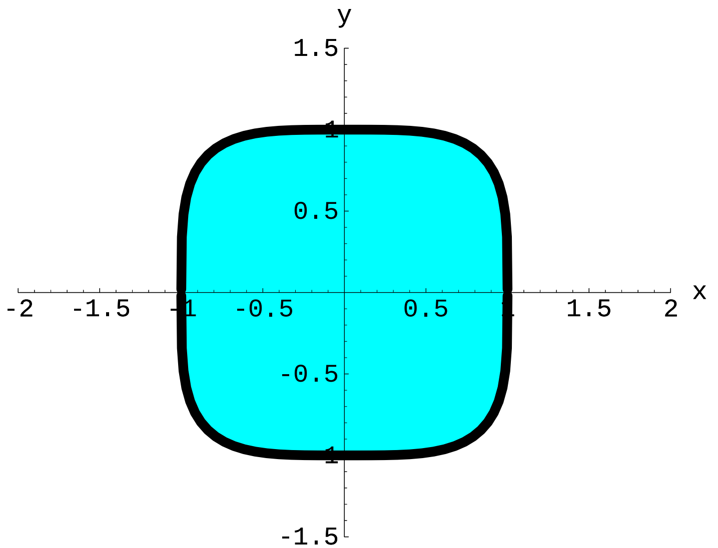


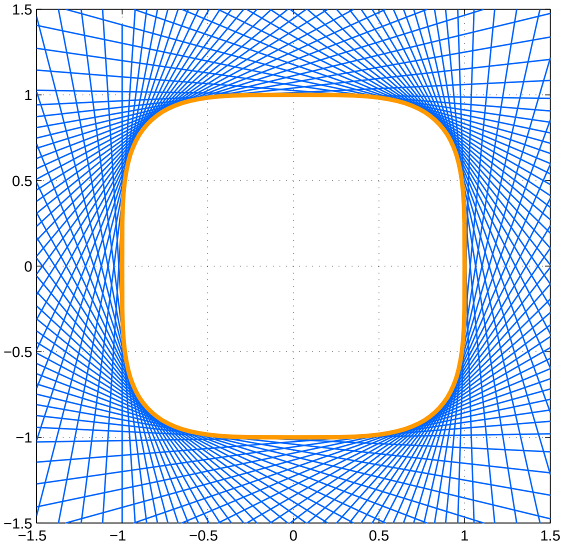


Consider the set described by $x^4 + y^4 \leq 1$

- Fails the rigid convexity condition.
- The SOS construction is exact.

Example





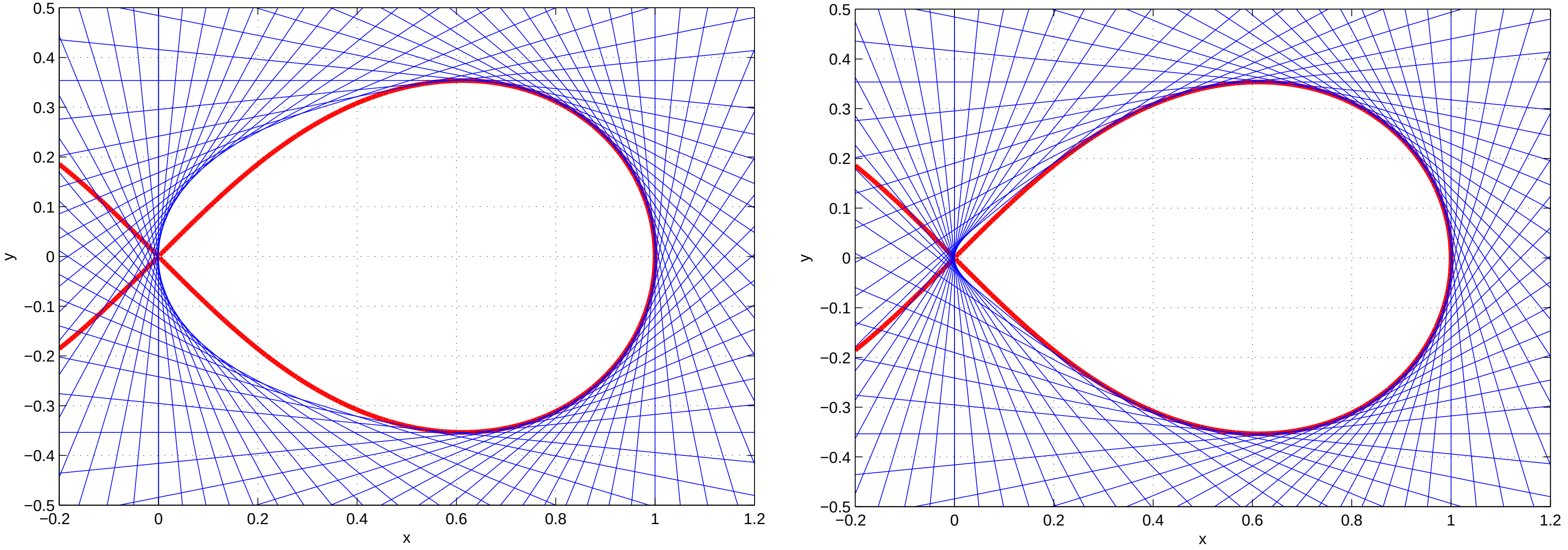
Consider the set described by $x^4 + y^4 \leq 1$

- Fails the rigid convexity condition.
- The SOS construction is exact.

Unfortunately, the SOS construction is *not* universal. Even if lifted SDP representations exist, it may fail. However, by the Helton-Nie theorem, this can happen only on the vanishing curvature case, or on singularities.

SOS fails

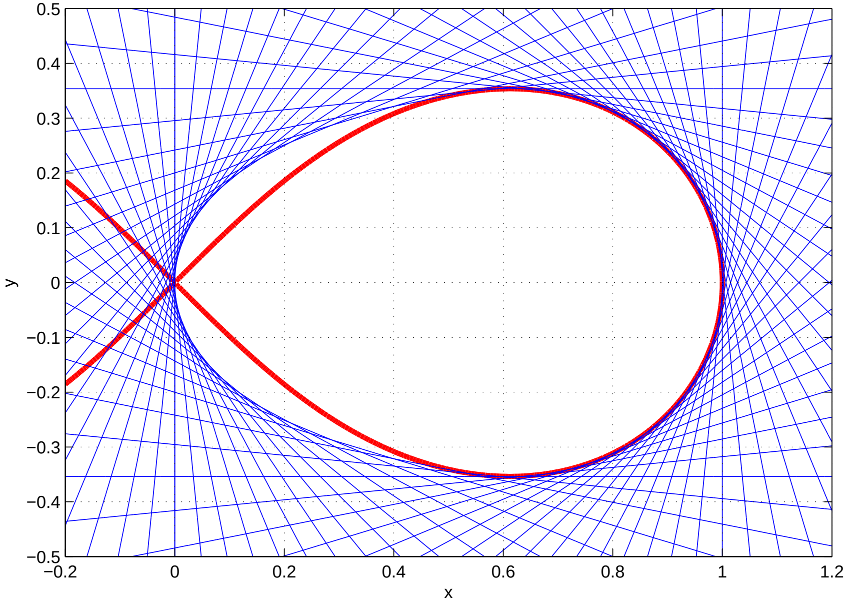
SOS schemes (Schmüdgen, Putinar/Lasserre) give outer approximations, but in this example they are *never* exact.

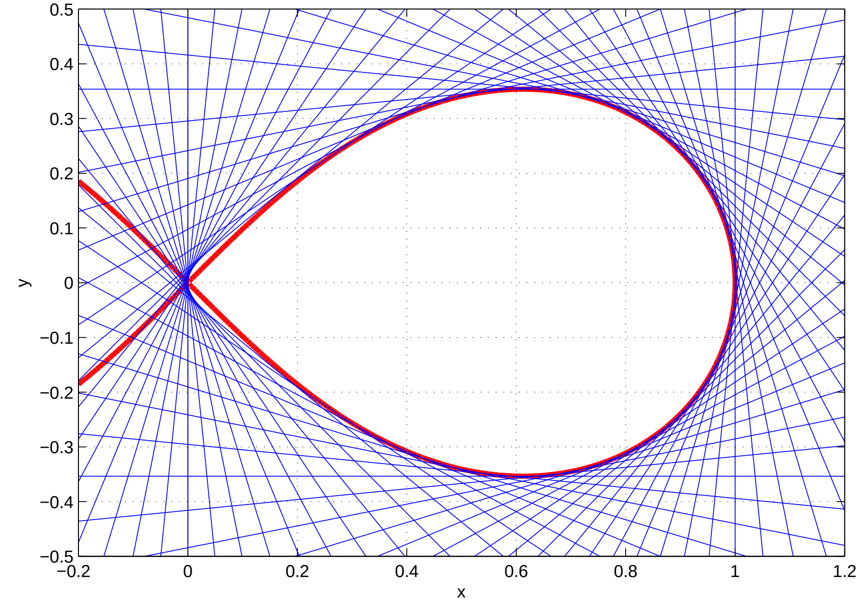


Massachusetts Institute of Technology

SOS fails

SOS schemes (Schmüdgen, Putinar/Lasserre) give outer approximations, but in this example they are *never* exact.

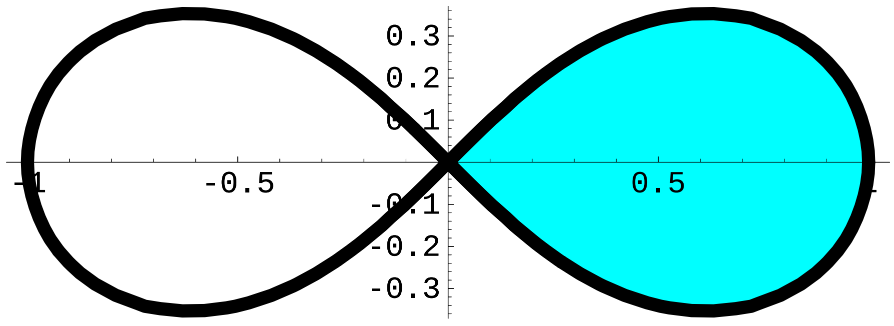




Can prove that this happens for *all* values of k.

Massachusetts Institute of Technology

SDP representation exist

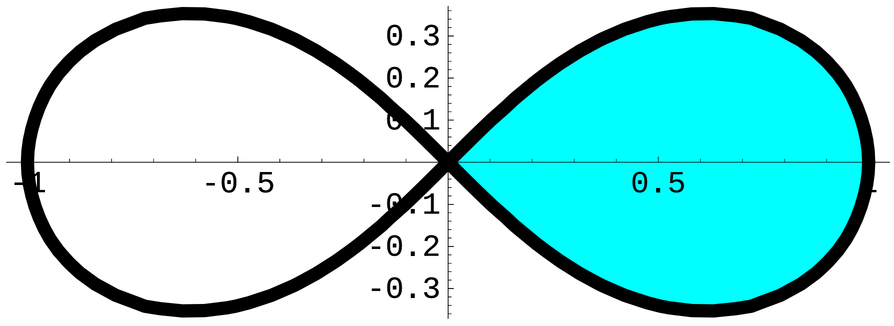


Thus, the set above can be represented as:

$$\begin{bmatrix} z & x & y \\ x & 1 & 0 \\ y & 0 & 1 \end{bmatrix} \succeq 0, \qquad \begin{bmatrix} x & y & z \\ y & x & 0 \\ z & 0 & x \end{bmatrix} \succeq 0.$$

Massachusetts Institute of Technology

SDP representation exist

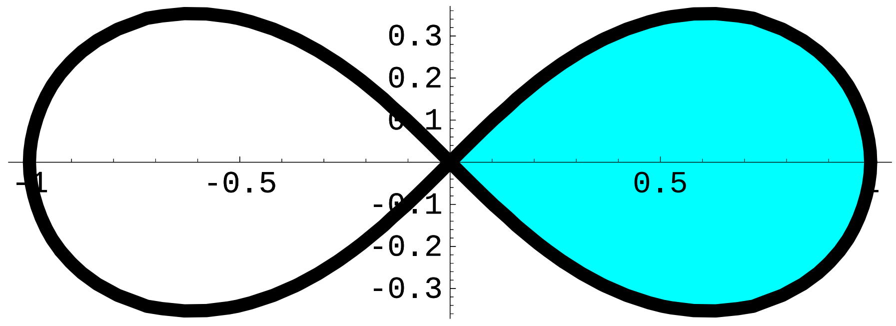


Thus, the set above can be represented as:

$$\begin{bmatrix} z & x & y \\ x & 1 & 0 \\ y & 0 & 1 \end{bmatrix} \succeq 0, \qquad \begin{bmatrix} x & y & z \\ y & x & 0 \\ z & 0 & x \end{bmatrix} \succeq 0.$$

In this case, We can obtain SDP representations in an algorithmic way, via an alternative P-satz construction.

SDP representation exist



Thus, the set above can be represented as:

$$\begin{bmatrix} z & x & y \\ x & 1 & 0 \\ y & 0 & 1 \end{bmatrix} \succeq 0, \qquad \begin{bmatrix} x & y & z \\ y & x & 0 \\ z & 0 & x \end{bmatrix} \succeq 0.$$

In this case, We can obtain SDP representations in an algorithmic way, via an alternative P-satz construction.

Still, not fully general. But, can do some cool examples...

Example: orthogonal matrices

Consider $O(3)$, the group of 3×3 orthogonal matrices of determinant one.

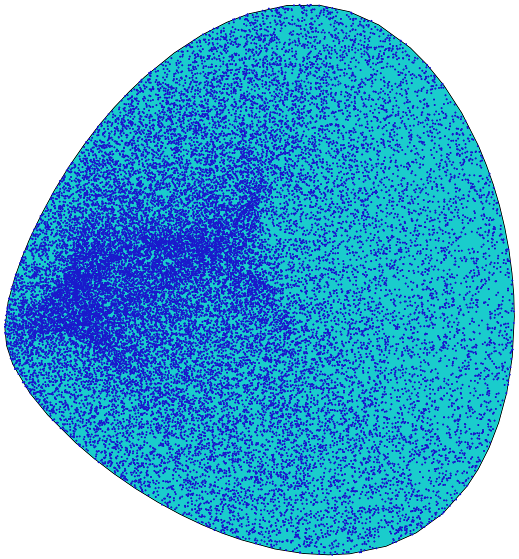
It has two connected components (sign of determinant).

We can use the double-cover of $SO(3)$ with $SU(2)$ to provide an exact SDP representation of the convex hull of $SO(3)$:

$$\begin{bmatrix} Z_{11}+Z_{22}-Z_{33}-Z_{44} & 2Z_{23}-2Z_{14} & 2Z_{24}+2Z_{13} \\ 2Z_{23}+2Z_{14} & Z_{11}-Z_{22}+Z_{33}-Z_{44} & 2Z_{34}-2Z_{12} \\ 2Z_{24}-2Z_{13} & 2Z_{34}+2Z_{12} & Z_{11}-Z_{22}-Z_{33}+Z_{44} \end{bmatrix}, \quad Z \succeq 0, \quad \operatorname{Tr} Z = 1.$$

This is a convex set in $\mathbb{R}^9$.

Here is a two-dimensional projection.



Polynomial games

Games with an *infinite* number of pure strategies.

In particular, strategy sets are semialgebraic, defined by polynomial equations and inequalities.

Simplest case (introduced by Dresher-Karlin-Shapley): two players, zero-sum, payoff given by $P(x, y)$, strategy space is a product of intervals.

Polynomial games

Games with an *infinite* number of pure strategies.

In particular, strategy sets are semialgebraic, defined by polynomial equations and inequalities.

Simplest case (introduced by Dresher-Karlin-Shapley): two players, zero-sum, payoff given by $P(x, y)$, strategy space is a product of intervals.

Reducible to a minimax problem, over convex hulls of algebraic varieties (the mixed strategies).

Polynomial games

Games with an *infinite* number of pure strategies.

In particular, strategy sets are semialgebraic, defined by polynomial equations and inequalities.

Simplest case (introduced by Dresher-Karlin-Shapley): two players, zero-sum, payoff given by $P(x, y)$, strategy space is a product of intervals.

Reducible to a minimax problem, over convex hulls of algebraic varieties (the mixed strategies).

Thm: (P.) The value of the game, and the corresponding optimal mixed strategies, can be computed by solving a SDP program.

Perfect generalization of the classical LP for finite games.

Related results for multiplayer games and correlated equilibria (w/ N. Stein and A. Ozdaglar).

Minimum rank and convex relaxations

Consider the rank minimization problem

$$\text{minimize}\ \operatorname{rank} X \qquad \text{subject to}\ \mathcal{A}(X) = b,$$

where $\mathcal{A} : \mathbb{R}^{m\times n} \to \mathbb{R}^p$ is a linear map.

Find the minimum-rank matrix in a given subspace. In general, NP-hard.

Minimum rank and convex relaxations

Consider the rank minimization problem

$$\text{minimize} \operatorname{rank} X \qquad \text{subject to } \mathcal{A}(X) = b,$$

where $\mathcal{A} : \mathbb{R}^{m \times n} \to \mathbb{R}^p$ is a linear map.

Find the minimum-rank matrix in a given subspace. In general, NP-hard.

Since rank is hard, let's use instead its *convex envelope*, the nuclear norm.

The nuclear norm of a matrix (alternatively, Schatten 1-norm, Ky Fan r-norm, or trace class norm) is the sum of its singular values, i.e.,

$$\|X\|_* := \sum_{i=1}^{r} \sigma_i(X).$$

Massachusetts
Institute of
Technology

Convex hulls and nuclear norm

Consider the unit ball of the nuclear norm $B := \{X \in \mathbb{R}^{m \times n} \, : \, ||X||_* \leq 1\}$.

Convex hull characterization:

$$B = \mathrm{conv}\{uv^T \, : \, u \in \mathbb{R}^m, v \in \mathbb{R}^n, ||u||^2 + ||v||^2 = 2\}$$

Exactly SDP-characterizable. Can solve the convex relaxation using SDP.

Massachusetts Institute of Technology

Convex hulls and nuclear norm

Consider the unit ball of the nuclear norm $B := \{X \in \mathbb{R}^{m\times n} \,:\, ||X||_* \leq 1\}$.

Convex hull characterization:

$$B = \mathrm{conv}\{uv^T \,:\, u \in \mathbb{R}^m, v \in \mathbb{R}^n, ||u||^2 + ||v||^2 = 2\}$$

Exactly SDP-characterizable. Can solve the convex relaxation using SDP.

Under certain conditions (e.g., if $\mathcal{A}$ is "random"), optimizing the nuclear norm yields the true minimum rank solution.

Connections to "compressed sensing".

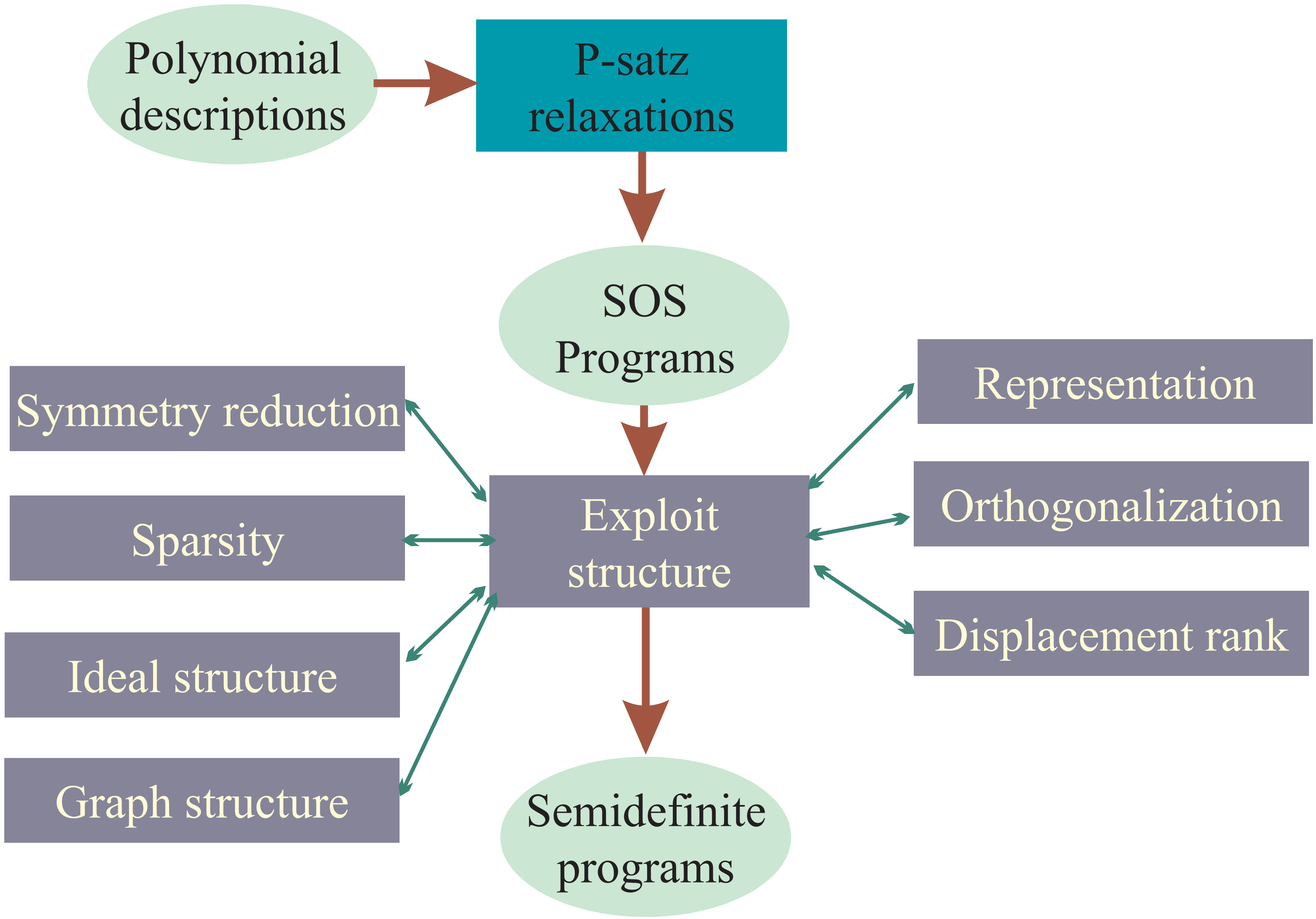
For details, see `arXiv:0706.4138`: Recht-Fazel-P., "Guaranteed Minimum-Rank Solutions of Linear Matrix Equations via Nuclear Norm Minimization" (2007)

Connections

Many fascinating links to other areas of mathematics:

- Probability (moments, exchangeability and de Finetti, etc)
- Operator theory (via Gelfand-Neimark-Segal)
- Harmonic analysis on semigroups
- Noncommutative probability (i.e., quantum mechanics)
- Complexity and proof theory (degrees of certificates)
- Graph theory (perfect graphs)
- Tropical geometry (SDP over more general fields)

Exploiting structure



Algebraic structure

- Sparseness: few nonzero coefficients.
  - Newton polytopes techniques.
- Ideal structure: equality constraints.
  - SOS on *quotient rings*.
  - Compute in the coordinate ring. Quotient bases.
- Graph structure:
  - Dependency graph among the variables.
- Symmetries: invariance under a group (w/ K. Gatermann)
  - SOS on *invariant rings*
  - Representation theory and invariant-theoretic methods.
  - Enabling factor in applications (e.g., Markov chains)

Numerical structure

- Rank one SDPs.
  - Dual coordinate change makes all constraints rank one
  - Efficient computation of Hessians and gradients
- Representations
  - Interpolation representation
  - Orthogonalization
- Displacement rank
  - Fast solvers for search direction

Related work

- Related basic work: N.Z. Shor, Nesterov, Lasserre, etc.
- Systems and control (Prajna, Rantzer, Hol-Scherer, etc.)
- Sparse optimization (Waki-Kim-Kojima-Muramatsu, Lasserre, Nie-Demmel, etc.)
- Approximation algorithms (de Klerk-Laurent-P.)
- Filter design (Alkire-Vandenberghe, Hachez-Nesterov, etc.)
- Stability number of graphs (Laurent, Peña, Rendl)
- Geometric theorem proving (P.-Peretz)
- Quantum information theory (Doherty-Spedalieri-P., Childs-Landahl-P.)
- Joint spectral radius (P.-Jadbabaie)

Summary

- A very rich class of optimization problems
- Methods have enabled many new applications
- Interplay of many branches of mathematics
- Structure must be exploited for reliability and efficiency
- Combination of numerical and algebraic techniques.

Summary

- A very rich class of optimization problems
- Methods have enabled many new applications
- Interplay of many branches of mathematics
- Structure must be exploited for reliability and efficiency
- Combination of numerical and algebraic techniques.

If you want to know more:

- Papers, slides, lecture notes, software, etc.: `www.mit.edu/~parrilo`
- Upcoming "SDP and convex algebraic geometry" NSF FRG website `www.math.washington.edu/~thomas/frg/frg.html` (Helton/P./Nie/Sturmfels/Thomas)

Summary

- A very rich class of optimization problems
- Methods have enabled many new applications
- Interplay of many branches of mathematics
- Structure must be exploited for reliability and efficiency
- Combination of numerical and algebraic techniques.

If you want to know more:

- Papers, slides, lecture notes, software, etc.: `www.mit.edu/~parrilo`
- Upcoming "SDP and convex algebraic geometry" NSF FRG website `www.math.washington.edu/~thomas/frg/frg.html` (Helton/P./Nie/Sturmfels/Thomas)

Thanks for your attention!

Massachusetts Institute of Technology