Semidefinite Programming Relaxations and Algebraic Optimization in Control

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Overview

- Mathematical and computational theory, and applications to combinatorial, non-convex and nonlinear problems
  - Semidefinite programming
  - Real algebraic geometry
  - Duality and certificates

Schedule

1. Convexity and duality
2. Quadratically constrained quadratic programming
3. From duality to algebra
4. Algebra and geometry
5. Sums of squares and semidefinite programming
6. Polynomials and duality; the Positivstellensatz

1. Convexity and Duality

- Formulation of optimization problems
- Engineering examples
- Convex sets and functions
- Convex optimization problems
- Standard problems: LP and SDP
- Feasibility problems
- Algorithms
- Certificates and separating hyperplanes
- Duality and geometry
- Examples: LP and and SDP
- Theorems of alternatives

Optimization Problems

A familiar problem

$$\begin{array}{rll} \text{minimize} & f_0(x) & \\ \text{subject to} & f_i(x) \leq 0 & \text{for all } i = 1, \ldots, m \\ & h_i(x) = 0 & \text{for all } i = 1, \ldots, p \end{array}$$

- $x \in \mathbb{R}^n$ is the variable
- $f_0 : \mathbb{R}^n \to \mathbb{R}$ is the *objective function*
- $f_i : \mathbb{R}^n \to \mathbb{R}$ for $i = 1, \ldots, m$ define *inequality constraints*
- $h_i : \mathbb{R}^n \to \mathbb{R}$ for $i = 1, \ldots, p$ define *equality constraints*

Discrete Problems: LQR with Binary Inputs

- linear discrete-time system $x(t+1) = Ax(t) + Bu(t)$ on interval $t = 0, \dots, N$

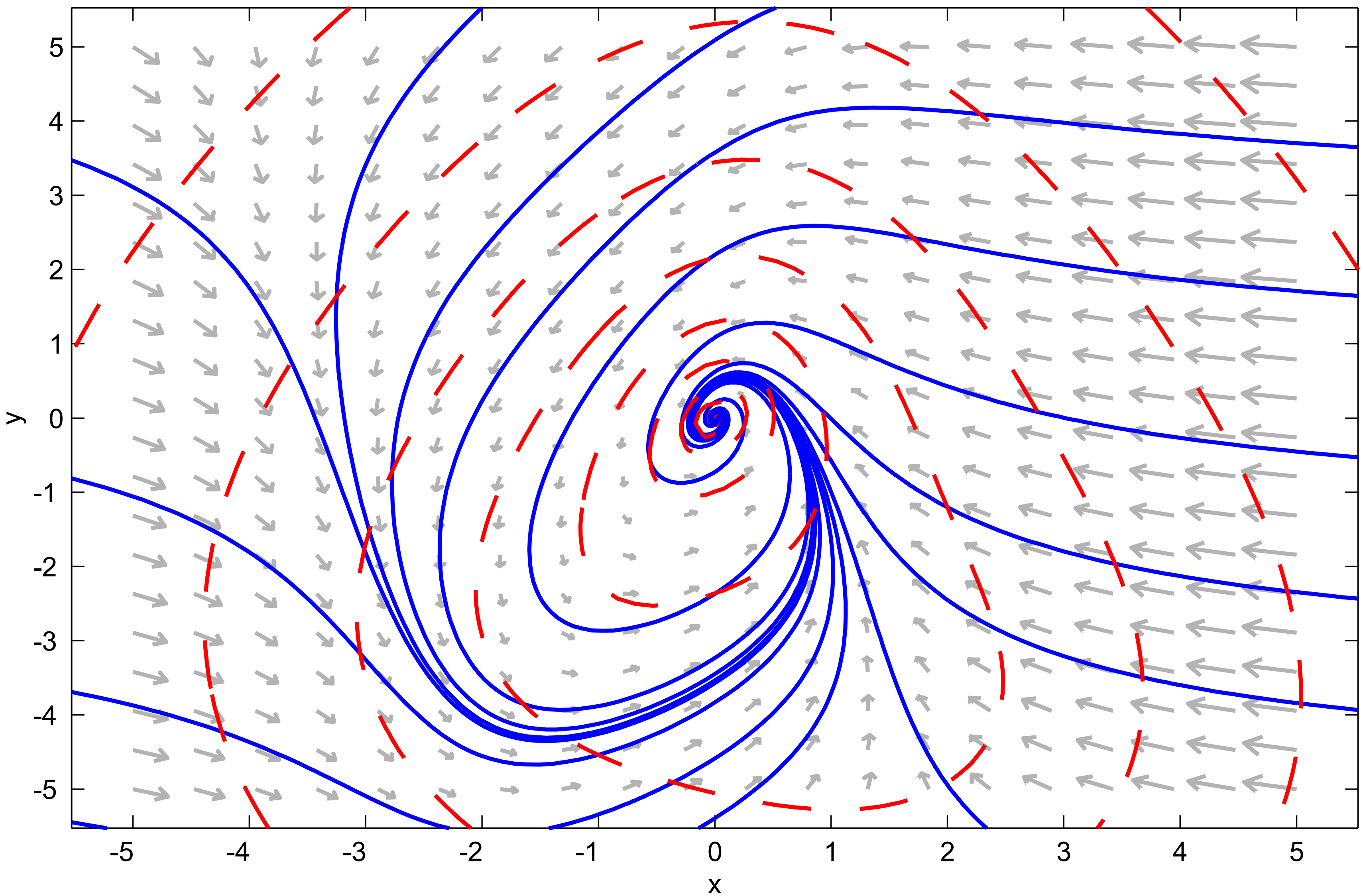
- objective is to minimize the quadratic tracking error

$$\sum_{t=0}^{N-1} \big(x(t) - r(t)\big)^T Q \big(x(t) - r(t)\big)$$

- using binary inputs

$$u_i(t) \in \{-1, 1\} \qquad \text{for all } i = 1, \dots, m, \text{ and } t = 0, \dots, N-1$$

Nonlinear Problems: Lyapunov Stability



Entanglement and Quantum Mechanics

- Entanglement is a behavior of quantum states, which cannot be explained classically.
- Responsible for many of the non-intuitive properties, and computational power of quantum devices.

A bipartite mixed quantum state ρ is *separable* (not *entangled*) if

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i| \otimes |\phi_i\rangle\langle\phi_i| \qquad \sum p_i = 1$$

for some ψ_i, ϕ_i.

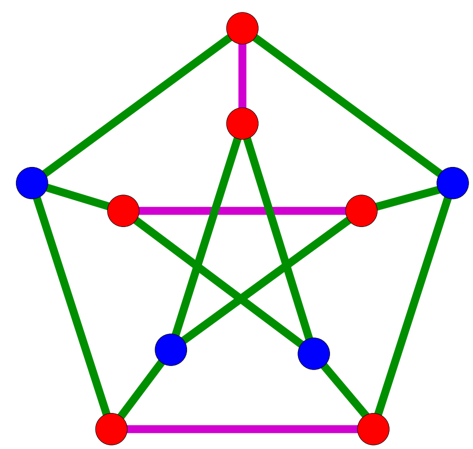
Given ρ, how to *decide* and *certify* if it is entangled?

Graph problems

Graph problems appear in many areas: MAX-CUT, independent set, cliques, etc.

MAX CUT partitioning

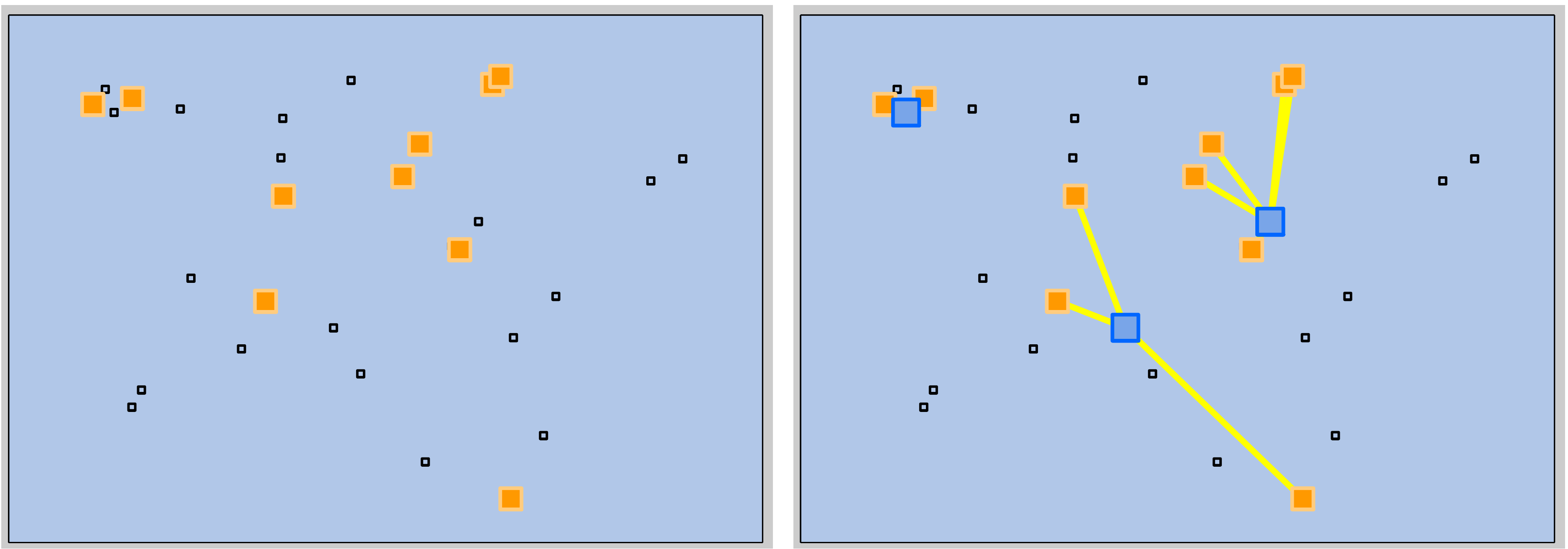
- Partition the nodes of a graph in two disjoint sets, maximizing the number of edges between sets.
- Practical applications (circuit layout, etc.)
- NP-complete.



How to compute bounds, or exact solutions, for this kind of problems?

Facility Location

- Given a set of n *cities*
- We'd like to open at most m facilities
- And assign each city to exactly one facility



Basic Nomenclature

A set $S \subset \mathbb{R}^n$ is called

- *affine* if $x, y \in S$ implies $\theta x + (1 - \theta)y \in S$ for all $\theta \in \mathbb{R}$; i.e., the line through x, y is contained in S
- *convex* if $x, y \in S$ implies $\theta x + (1 - \theta)y \in S$ for all $\theta \in [0, 1]$; i.e., the line segment between x and y is contained in S.
- *a convex cone* if $x, y \in S$ implies $\lambda x + \mu y \in S$ for all $\lambda, \mu \geq 0$; i.e., the *pie slice* between x and y is contained in S.

A function $f : \mathbb{R}^n \to \mathbb{R}$ is called

- *affine* if $f(\theta x + (1 - \theta)y) = \theta f(x) + (1 - \theta)f(y)$ for all $\theta \in \mathbb{R}$ and $x, y \in \mathbb{R}^n$; i.e., f is equals a linear function plus a constant $f = Ax + b$
- *convex* if $f\big(\theta x + (1 - \theta)y\big) \leq \theta f(x) + (1 - \theta)f(y)$ for all $\theta \in [0, 1]$ and $x, y \in \mathbb{R}^n$

Properties of Convex Functions

- $f_1 + f_2$ is convex if f_1 and f_2 are
- $f(x) = \max\{f_1(x), f_2(x)\}$ is convex if f_1 and f_2 are
- $g(x) = \sup_y f(x, y)$ is convex if $f(x, y)$ is convex in x for each y
- convex functions are continuous on the interior of their domain
- $f(Ax + b)$ is convex if f is
- $Af(x) + b$ is convex if f is
- $g(x) = \inf_y f(x, y)$ is convex if $f(x, y)$ is jointly convex
- the $\alpha-$*sublevel set*

$$\{ x \in \mathbb{R}^n \mid f(x) \leq \alpha \}$$

is convex if f is convex; (the converse is not true)

Convex Optimization Problems

$$\begin{array}{lll} \text{minimize} & f_0(x) & \\ \text{subject to} & f_i(x) \leq 0 & \text{for all } i = 1, \ldots, m \\ & h_i(x) = 0 & \text{for all } i = 1, \ldots, p \end{array}$$

This problem is called a *convex program* if

- the objective function f_0 is convex
- the inequality constraints f_i are convex
- the equality constraints h_i are affine

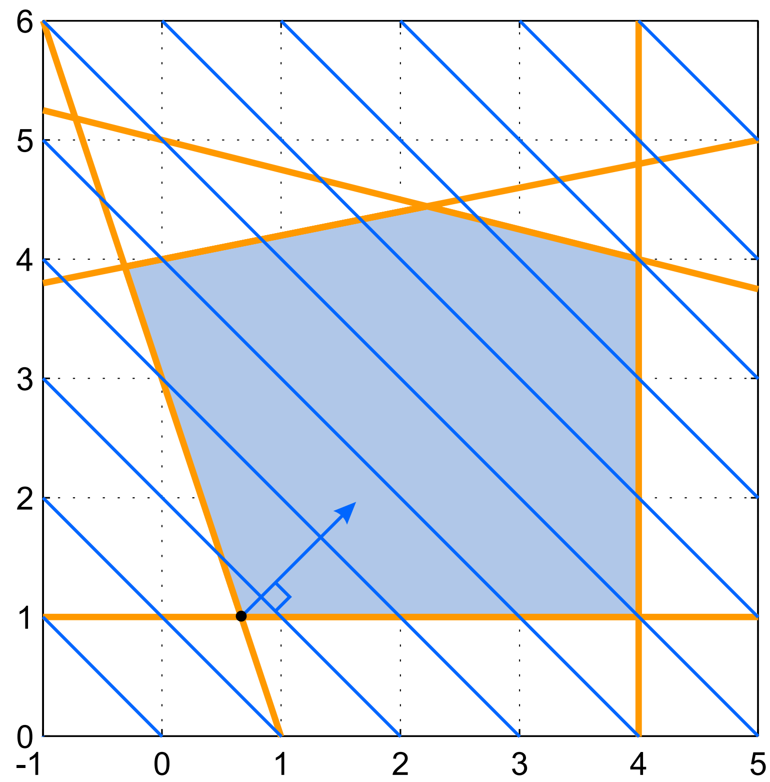
Linear Programming (LP)

In a *linear program*, the objective and constraint functions are affine.

$$\begin{array}{rl} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & Cx \leq d \end{array}$$

Example

$$\begin{array}{rr} \text{minimize} & x_1 + x_2 \\ \text{subject to} & 3x_1 + x_2 \geq 3 \\ & x_2 \geq 1 \\ & x_1 \leq 4 \\ & -x_1 + 5x_2 \leq 20 \\ & x_1 + 4x_2 \leq 20 \end{array}$$



Linear Programming

Every linear program may be written in the *standard primal form*

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{subject to} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

Here $x \in \mathbb{R}^n$, and $x \geq 0$ means $x_i \geq 0$ for all i

- The *nonnegative orthant* $\{ x \in \mathbb{R}^n \mid x \geq 0 \}$ is a *convex cone*.
- This convex cone defines the partial ordering $\geq$ on $\mathbb{R}^n$
- Geometrically, the feasible set is the intersection of an affine set with a convex cone.

Semidefinite Programming

$$\begin{array}{lll} \text{minimize} & \mathbf{trace}\, CX & \\ \text{subject to} & \mathbf{trace}\, A_i X = b_i & \text{for all } i = 1, \dots, m \\ & X \succeq 0 & \end{array}$$

- The variable X is in the set of $n \times n$ symmetric matrices

$$\mathbb{S}^n = \{ A \in \mathbb{R}^{n \times n} \mid A = A^T \}$$

- $X \succeq 0$ means X is positive semidefinite
- As for LP, the feasible set is the intersection of an affine set with a convex cone, in this case the *positive semidefinite cone*

$$\{ X \in \mathbb{S}^n \mid X \succeq 0 \}$$

Hence the feasible set is convex.

SDPs with Explicit Variables

We can also explicitly parametrize the affine set to give

$$\begin{array}{rl} \text{minimize} & c^T x \\ \text{subject to} & F_0 + x_1 F_1 + x_2 F_2 + \cdots + x_n F_n \preceq 0 \end{array}$$

where $F_0, F_1, \ldots, F_n$ are symmetric matrices.

The inequality constraint is called a *linear matrix inequality*; e.g.,

$$\begin{bmatrix} x_1 - 3 & x_1 + x_2 & -1 \\ x_1 + x_2 & x_2 - 4 & 0 \\ -1 & 0 & x_1 \end{bmatrix} \preceq 0$$

which is equivalent to

$$\begin{bmatrix} -3 & 0 & -1 \\ 0 & -4 & 0 \\ -1 & 0 & 0 \end{bmatrix} + x_1 \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} + x_2 \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \preceq 0$$

The Feasible Set is Semialgebraic

The *(basic closed) semialgebraic set* defined by polynomials $f_1, \ldots, f_m$ is

$$\Big\{ x \in \mathbb{R}^n \mid f_i(x) \geq 0 \text{ for all } i = 1, \ldots, m \Big\}$$

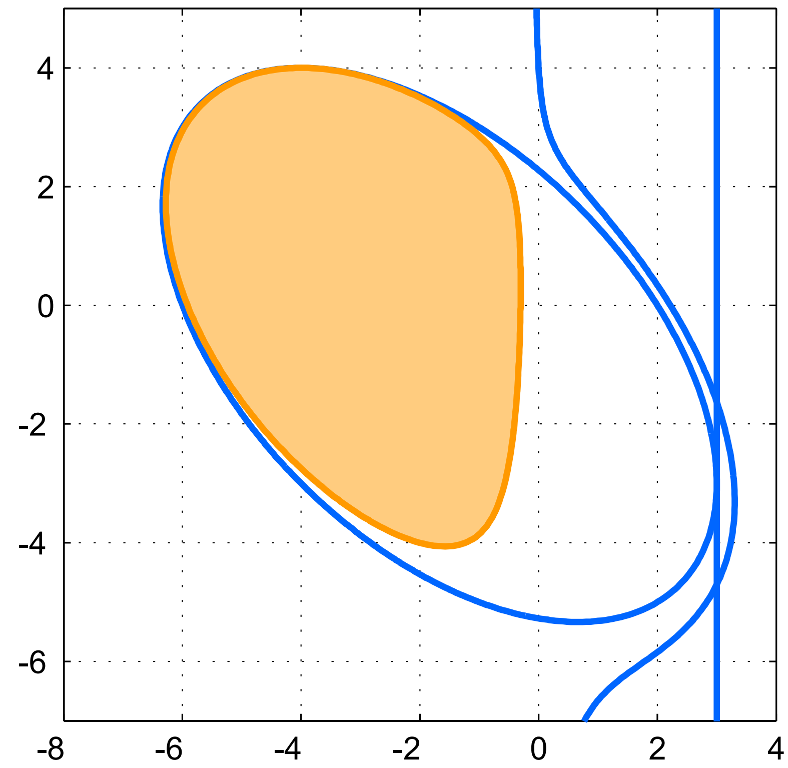
The feasible set of an SDP is a semialgebraic set.

Because a matrix $A \succ 0$ if and only if

$$\det(A_k) > 0 \text{ for } k = 1, \ldots, n$$

where A_k is the submatrix of A consisting of the first k rows and columns.

The Feasible Set



For example

$$0 \prec \begin{bmatrix} 3 - x_1 & -(x_1 + x_2) & 1 \\ -(x_1 + x_2) & 4 - x_2 & 0 \\ 1 & 0 & -x_1 \end{bmatrix}$$

is equivalent to the polynomial inequalities

$$0 < 3 - x_1$$

$$0 < (3 - x_1)(4 - x_2) - (x_1 + x_2)^2$$

$$0 < -x_1((3 - x_1)(4 - x_2) - (x_1 + x_2)^2) - (4 - x_2)$$

Intersection of Feasible Sets

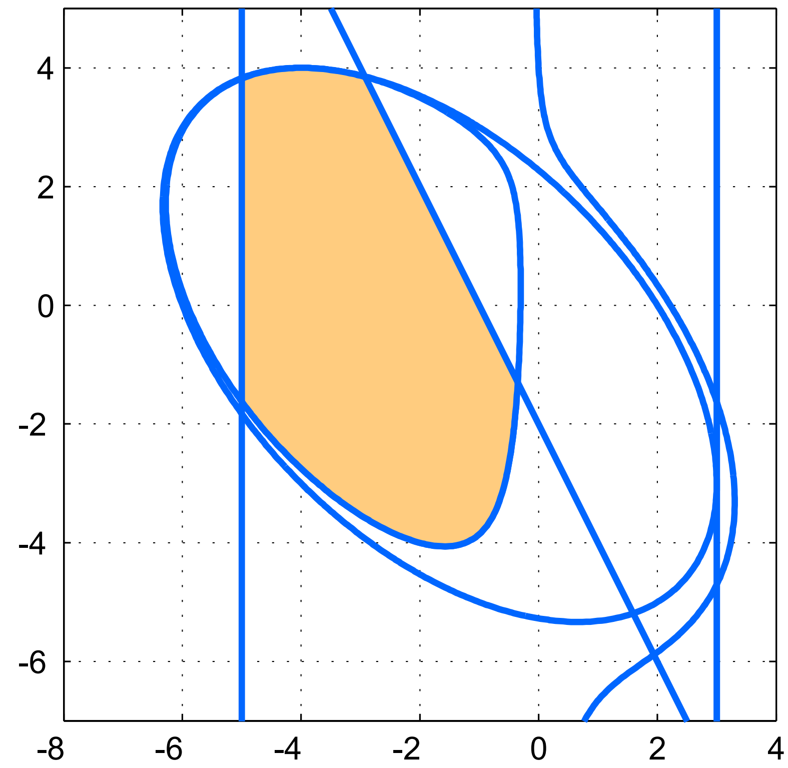
The intersection of the feasible sets

$$\begin{bmatrix} 2x_1 + x_2 + 2 & 0 \\ 0 & -x_1 - 5 \end{bmatrix} \prec 0$$

and

$$\begin{bmatrix} x_1 - 3 & x_1 + x_2 & -1 \\ x_1 + x_2 & x_2 - 4 & 0 \\ -1 & 0 & x_1 \end{bmatrix} \prec 0$$

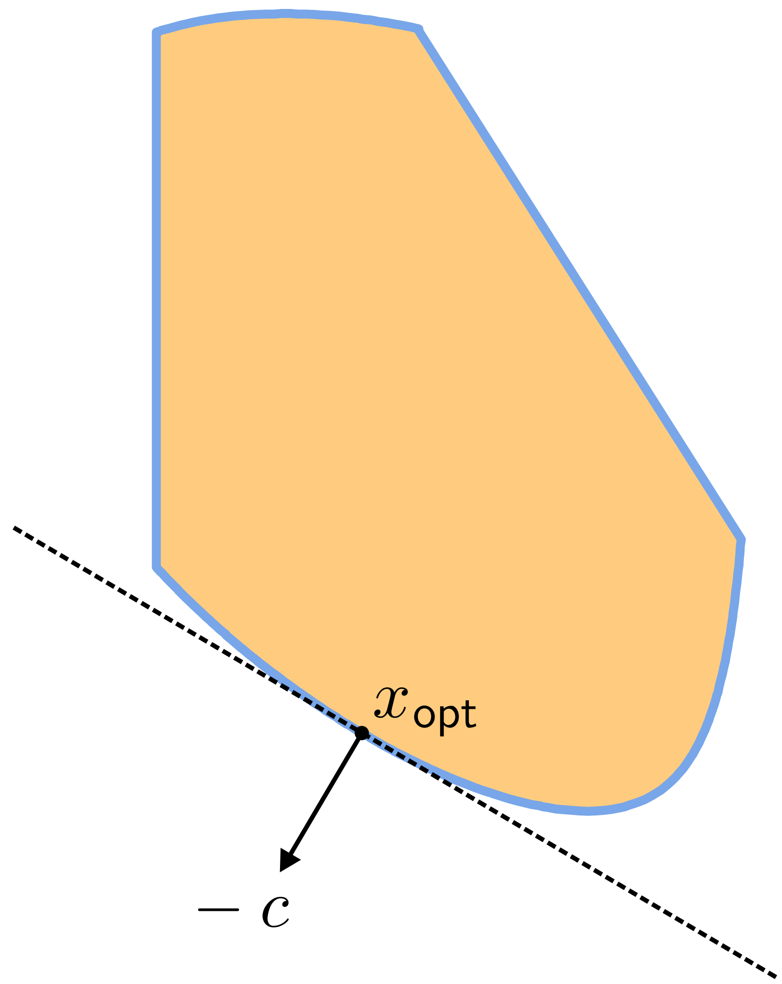
is given by



$$\begin{bmatrix} x_1 - 3 & x_1 + x_2 & -1 & 0 & 0 \\ x_1 + x_2 & x_2 - 4 & 0 & 0 & 0 \\ -1 & 0 & x_1 & 0 & 0 \\ 0 & 0 & 0 & 2x_1 + x_2 + 2 & 0 \\ 0 & 0 & 0 & 0 & -x_1 - 5 \end{bmatrix} \prec 0$$

Optimal Points

Since SDPs are convex, if the feasible set is closed then the optimal is always achieved on the boundary.



Convex Optimization Problems

For a convex optimization problem, the *feasible set*

$$S = \{\, x \in \mathbb{R}^n \mid f_i(x) \le 0 \text{ and } h_j(x) = 0 \text{ for all } i, j \,\}$$

is convex. So we can write the problem as

$$\begin{array}{rl} \text{minimize} & f_0(x) \\ \text{subject to} & x \in S \end{array}$$

This approach emphasizes the *geometry* of the problem.

For a convex optimization problem, any local minimum is also a global minimum.

Feasibility Problems

We are also interested in *feasibility problems* as follows. Does there exist $x \in \mathbb{R}^n$ which satisfies

$$\begin{aligned} f_i(x) &\leq 0 && \text{for all } i = 1, \ldots, m \\ h_i(x) &= 0 && \text{for all } i = 1, \ldots, p \end{aligned}$$

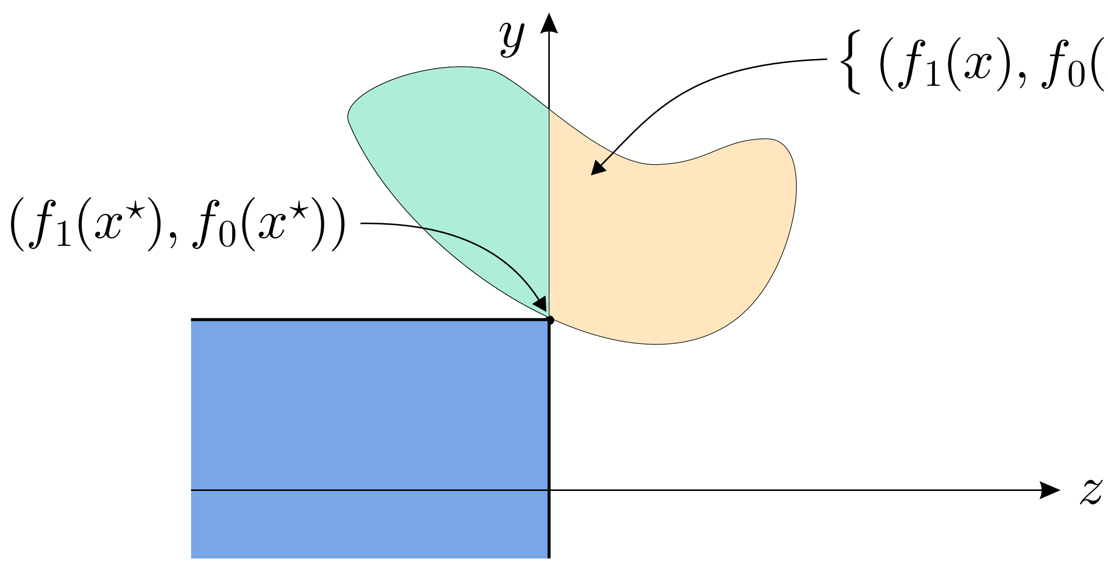
If there does not exist such an x, the problem is described as *infeasible*.

Feasibility Problems

We can always convert an optimization problem into a feasibility problem; does there exist $x \in \mathbb{R}^n$ such that

$$f_0(x) \leq t$$
$$f_i(x) \leq 0$$
$$h_i(x) = 0$$

Bisection search over the parameter t finds the optimal.



Feasibility Problems

Conversely, we can convert feasibility problems into optimization problems.

e.g. the feasibility problem of finding x such that

$$f_i(x) \leq 0 \qquad \text{for all } i = 1, \ldots, m$$

can be solved as

$$\begin{array}{ll} \text{minimize} & y \\ \text{subject to} & f_i(x) \leq y \qquad \text{for all } i = 1, \ldots, m \end{array}$$

where there are $n + 1$ variables $x \in \mathbb{R}^n$ and $y \in \mathbb{R}$

This technique may be used to find an initial feasible point for optimization algorithms

Algorithms

For convex optimization problems, there are several good algorithms

- interior-point algorithms work well in theory and practice
- for certain classes of problems, (e.g. LP and SDP) there is a worst-case time-complexity bound
- conversely, some convex optimization problems are known to be NP-hard
- problems are specified either in *standard form*, for LPs and SDPs, or via an *oracle*

Certificates

Consider the feasibility problem

Does there exist $x \in \mathbb{R}^n$ which satisfies

$$f_i(x) \leq 0 \text{ for all } i = 1, \dots, m$$
$$h_i(x) = 0 \text{ for all } i = 1, \dots, p$$

There is a fundamental asymmetry between establishing that

- There exists at least one feasible x
- The problem is infeasible

To show existence, one needs a *feasible point* $x \in \mathbb{R}^n$.

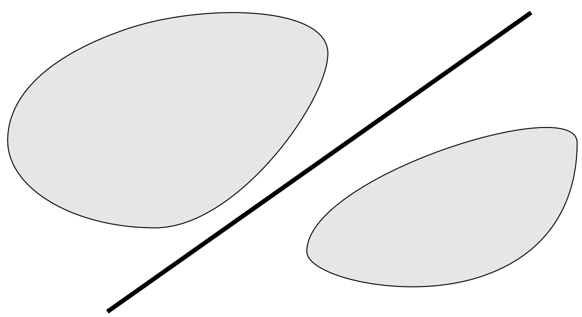
To show emptiness, one needs a a *certificate of infeasibility*; a mathematical proof that the problem is infeasible.

Certificates and Separating Hyperplanes

The simplest form of certificate is a *separating hyperplane*. The idea is that a hyperplane $L \subset \mathbb{R}^n$ breaks $\mathbb{R}^n$ into two half-spaces,

$$H_1 = \left\{ x \in \mathbb{R}^n \mid b^T x \le a \right\} \qquad \text{and} \qquad H_2 = \left\{ x \in \mathbb{R}^n \mid b^T x > a \right\}$$

If two *closed convex* sets are disjoint, there is a hyperplane that separates them.



So to prove infeasibility of

$$f_i(x) \le 0 \qquad \text{for } i = 1, 2$$

we show that

$$\{ x \in \mathbb{R}^n \mid f_1(x) \le 0 \} \subset H_1 \qquad \text{and} \qquad \{ x \in \mathbb{R}^n \mid f_2(x) \le 0 \} \subset H_2$$

Duality

We'd like to solve

$$\begin{array}{rll} \text{minimize} & f_0(x) & \\ \text{subject to} & f_i(x) \leq 0 & \text{for all } i = 1, \dots, m \\ & h_i(x) = 0 & \text{for all } i = 1, \dots, p \end{array}$$

define the *Lagrangian* for $x \in \mathbb{R}^n$, $\lambda \in \mathbb{R}^m$ and $\nu \in \mathbb{R}^p$ by

$$L(x, \lambda, \nu) = f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \nu_i h_i(x)$$

and the *Lagrange dual function*

$$g(\lambda, \nu) = \inf_{x \in \mathbb{R}^n} L(x, \lambda, \nu)$$

We allow $g(\lambda, \nu) = -\infty$ when there is no finite infimum

Duality

the *dual problem* is

$$\begin{array}{rl} \text{maximize} & g(\lambda, \nu) \\ \text{subject to} & \lambda \geq 0 \end{array}$$

we call λ, ν *dual feasible* if $\lambda \geq 0$ and $g(\lambda, \nu)$ is finite.

- The dual function g is always concave, even if the primal problem is not convex

Weak Duality

For any primal feasible x and dual feasible λ, ν we have

$$g(\lambda, \nu) \leq f_0(x)$$

because

$$\begin{aligned} g(\lambda, \nu) &\leq f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \nu_i h_i(x) \\ &\leq f_0(x) \end{aligned}$$

- A feasible λ, ν provides a *certificate* that the primal optimal is greater than $g(\lambda, \nu)$
- many interior-point methods simultaneously optimize the primal and the dual problem; when $f_0(x) - g(\lambda, \nu) \leq \varepsilon$ we know that x is $\varepsilon-$suboptimal

Strong Duality

- $p^\star$ is the optimal value of the primal problem,
- $d^\star$ is the optimal value of the dual problem

Weak duality means $p^\star \geq d^\star$

If $p^\star = d^\star$ we say *strong duality* holds. Equivalently, we say the *duality gap* $p^\star - d^\star$ is zero.

Constraint qualifications give sufficient conditions for strong duality.

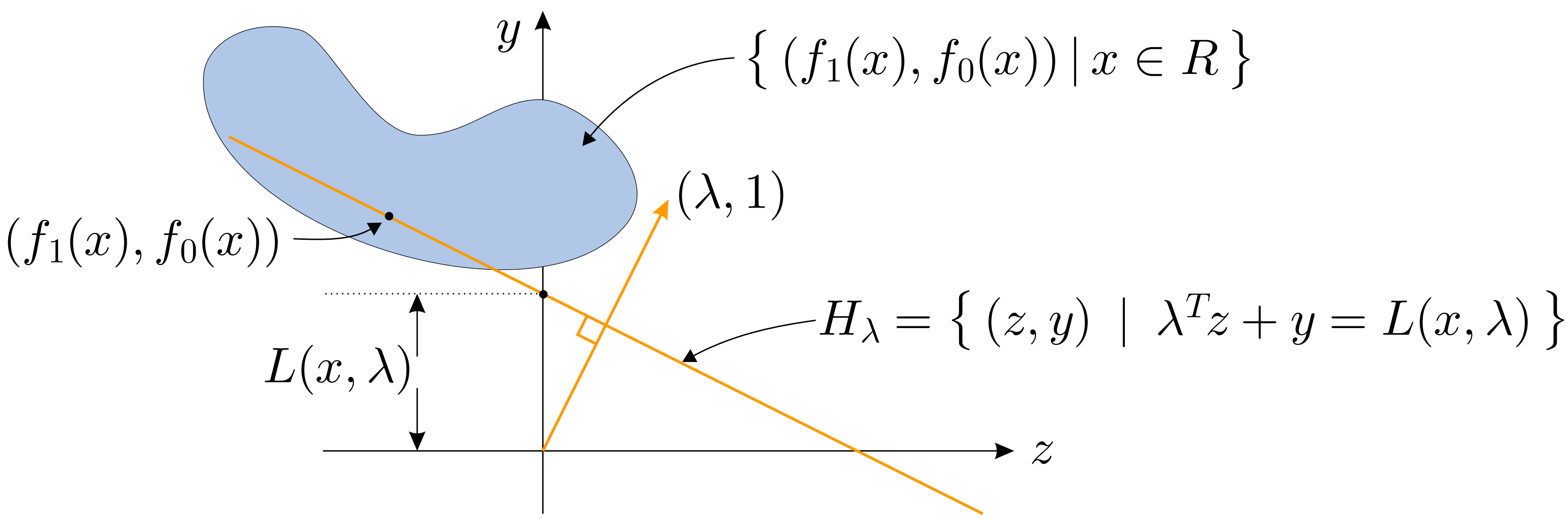
An example is *Slater's condition*; strong duality holds if the primal problem is convex and strictly feasible.

Geometric Interpretations: The Lagrangian

consider the optimization problem

$$\begin{aligned} \text{minimize} \quad & f_0(x) \\ \text{subject to} \quad & f_1(x) \leq 0 \end{aligned}$$

The value of the Lagrangian $L(x, \lambda)$ is the intersection of the hyperplane H_λ with the vertical axis

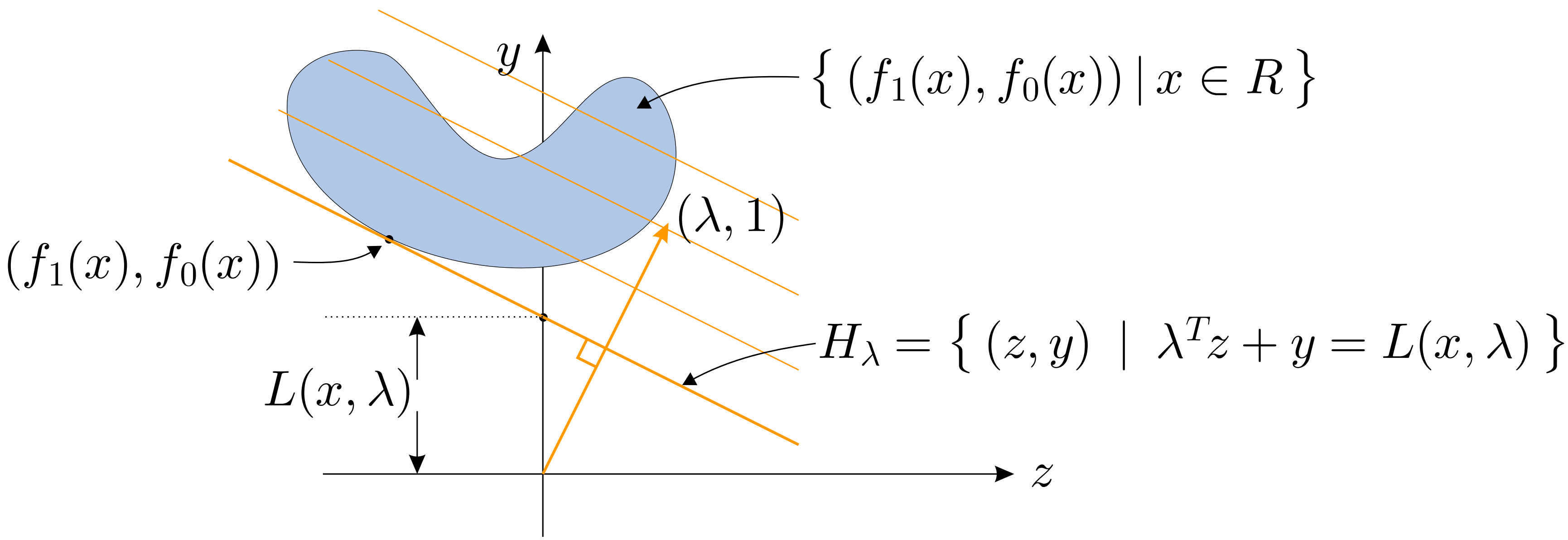


The Lagrange Dual Function

The Lagrange dual function is

$$g(\lambda) = \inf_{x \in \mathbb{R}^n} L(x, \lambda)$$

i.e., the minimum intersection for a given slope $-\lambda$



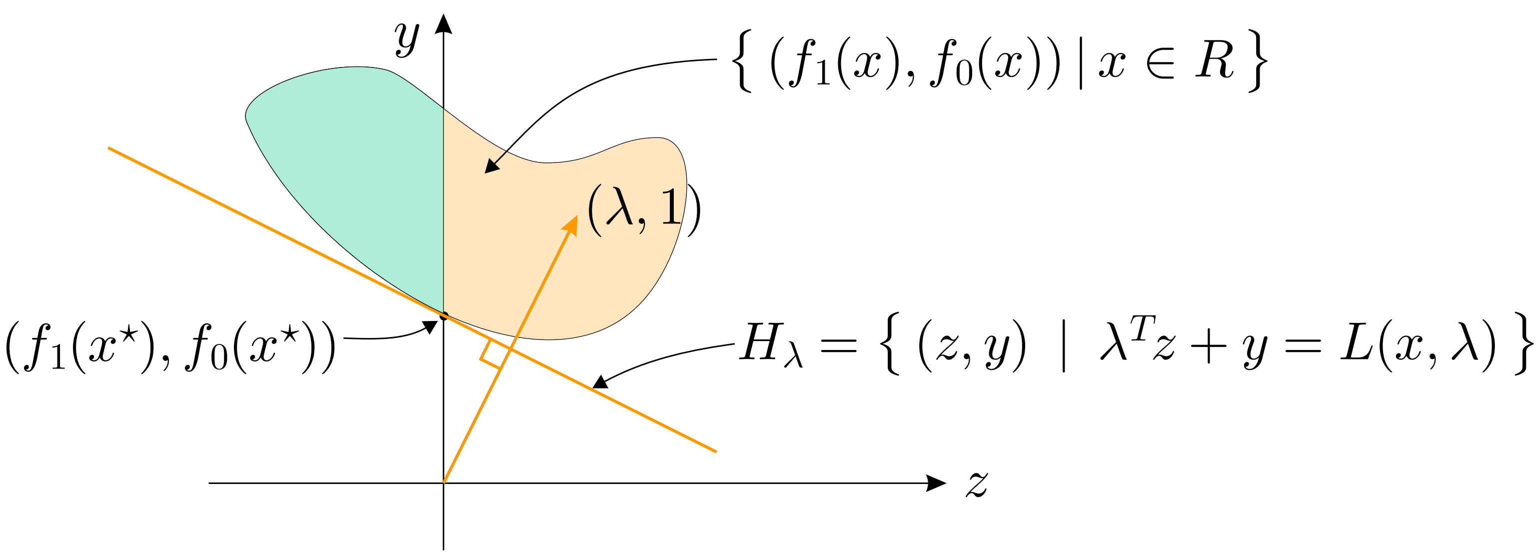
Sensitivity

consider the perturbed problem

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \le y_i \quad \text{for all } i = 1, \dots, m \end{array}$$

and let $p^\star(y)$ be the optimal value parametrized by y. Then for any optimal $\lambda^\star$ we have

$$\lambda^\star = -\nabla p^\star(0)$$

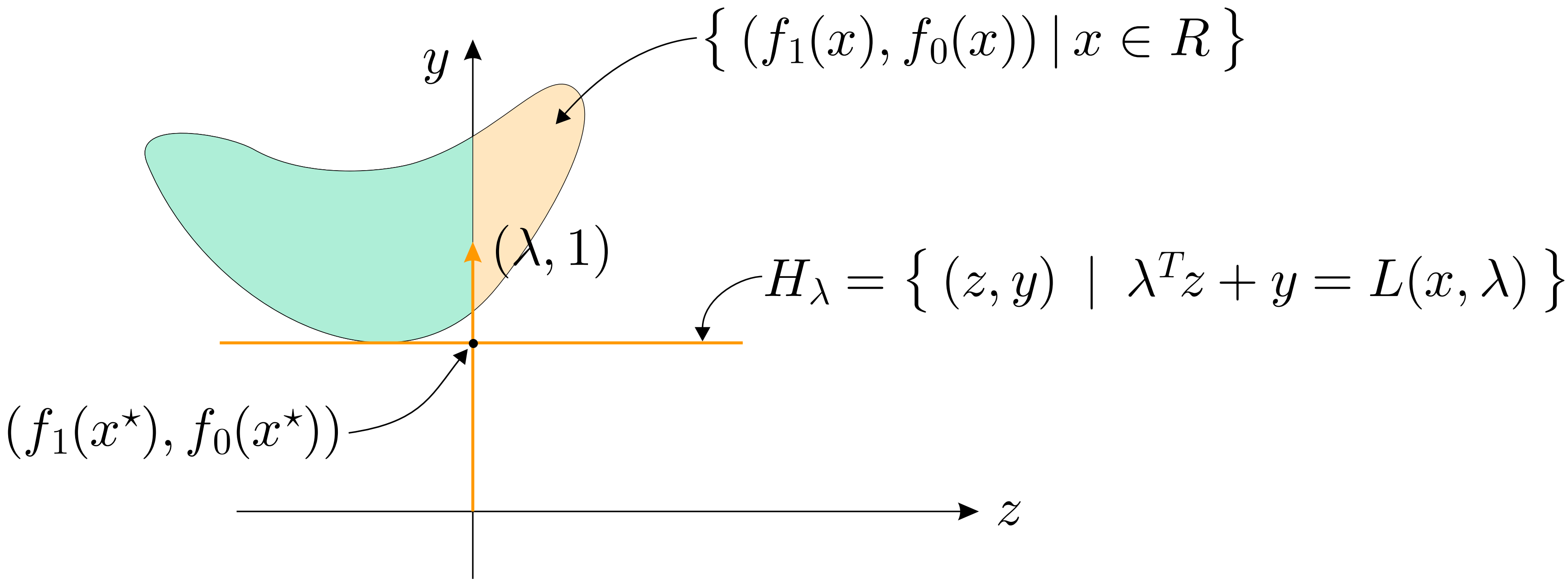


Complementary Slackness

For $\lambda^\star$ dual optimal, and $x^\star$ primal optimal, we have

$$\lambda_i^\star f_i(x^\star) = 0 \qquad \text{for all } i = 1, \dots, m$$

whenever strong duality holds; i.e., if the i'th constraint is active, then $\lambda_i^\star > 0$



Example: Linear Programming

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array}$$

The Lagrange dual function is

$$\begin{aligned} g(\lambda, \nu) &= \inf_{x \in \mathbb{R}^n} \Big(c^T x + \nu^T (b - Ax) - \lambda^T x \Big) \\ &= \begin{cases} b^T \nu & \text{if } c - A^T \nu - \lambda = 0 \\ -\infty & \text{otherwise} \end{cases} \end{aligned}$$

So the dual problem is

$$\begin{array}{ll} \text{maximize} & b^T \nu \\ \text{subject to} & A^T \nu \leq c \end{array}$$

Example: Semidefinite Programming

$$\begin{array}{lll} \text{minimize} & \mathbf{trace}\, CX & \\ \text{subject to} & \mathbf{trace}\, A_i X = b_i & \text{for all } i = 1, \dots, m \\ & X \succeq 0 & \end{array}$$

The Lagrange dual is

$$g(Z, \nu) = \inf_X \left(\mathbf{trace}\, CX - \mathbf{trace}\, ZX + \sum_{i=1}^m \nu_i (b_i - \mathbf{trace}\, A_i X) \right)$$

$$= \begin{cases} b^T \nu & \text{if } C - Z - \sum_{i=1}^m \nu_i A_i = 0 \\ -\infty & \text{otherwise} \end{cases}$$

So the dual problem is to maximize $b^T \nu$ subject to

$$C - Z - \sum_{i=1}^m \nu_i A_i = 0 \qquad \text{and} \qquad Z \succeq 0$$

Semidefinite Programming Duality

The primal problem is

$$\begin{array}{lll} \text{minimize} & \mathbf{trace}\, CX & \\ \text{subject to} & \mathbf{trace}\, A_i X = b_i & \text{for all } i = 1, \dots, m \\ & X \succeq 0 & \end{array}$$

The dual problem is

$$\begin{array}{ll} \text{maximize} & b^T \nu \\ \text{subject to} & \displaystyle\sum_{i=1}^{m} \nu_i A_i \preceq C \end{array}$$

The Fourfold Way

There are several ways of formulating an SDP for its numerical solution.

Because *subspaces* can be described

- Using *generators* or a *basis*; Equivalently, the subspace is the range of a linear map $\{ x \mid x = B\lambda \text{ for some } \lambda \}$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \lambda_1 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + \lambda_2 \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} \lambda_1 \\ \lambda_1 - \lambda_2 \\ 2\lambda_1 + 2\lambda_2 \end{bmatrix}$$

- Through the defining equations; i.e, as the *kernel* $\{ x \mid Ax = 0 \}$

$$\{(x_1, x_2, x_3) \in \mathbb{R}^3 \quad | \quad 4x_1 - 2x_2 - x_3 = 0\}$$

Depending on which description we use, and whether we write a primal or dual formulation, we have *four* possibilities (two primal-dual pairs).

Example: Two Primal-Dual Pairs

$$\begin{array}{ll} \text{maximize} & 2x + 2y \\ \text{subject to} & \begin{bmatrix} 1+x & y \\ y & 1-x \end{bmatrix} \succeq 0 \end{array}$$

$$\begin{array}{ll} \text{minimize} & \textbf{trace} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} W \\ \text{subject to} & \textbf{trace} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} W = 2 \\ & \textbf{trace} \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} W = 2 \\ & W \succeq 0 \end{array}$$

Another, *more efficient* fomulation which solves the same problem:

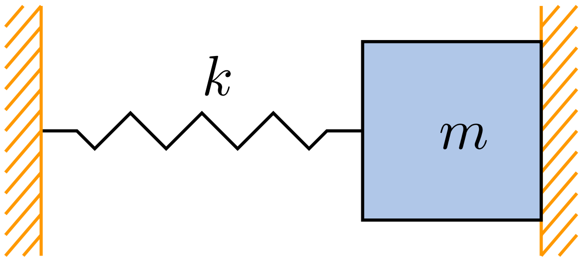
$$\begin{array}{ll} \text{maximize} & \textbf{trace} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} Z \\ \text{subject to} & \textbf{trace} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} Z = 2 \\ & Z \succeq 0 \end{array}$$

$$\begin{array}{ll} \text{minimize} & 2t \\ \text{subject to} & \begin{bmatrix} t-1 & -1 \\ -1 & t+1 \end{bmatrix} \succeq 0 \end{array}$$

Duality

- Duality has many interpretations; via economics, game-theory, geometry.
- e.g., one may interpret Lagrange multipliers as a price for violating constraints, which may correspond to resource limits or capacity constraints.
- Often physical problems associate specific meaning to certain Lagrange multipliers, e.g. pressure, momentum, force can all be viewed as Lagrange multipliers

Example: Mechanics



- Spring under compression
- Mass at horizontal position x, equilibrium at $x = 2$

$$\begin{array}{ll} \text{minimize} & \frac{k}{2}(x-2)^2 \\ \text{subject to} & x \leq 1 \end{array}$$

The Lagrangian is $L(x, \lambda) = \frac{k}{2}(x-2)^2 + \lambda(x-1)$

If λ is dual optimal and x is primal optimal, then $\frac{\partial}{\partial x} L(x, \lambda) = 0$, i.e.,

$$k(x-2) + \lambda = 0$$

so we can interpret λ as a *force*

Feasibility of Inequalities

The *primal feasibility problem* is

does there exist $x \in \mathbb{R}^n$ such that

$f_i(x) \geq 0 \qquad$ for all $i = 1, \dots, m$

The *dual function* $g : \mathbb{R}^m \to \mathbb{R}$ is

$$g(\lambda) = \sup_{x \in \mathbb{R}^n} \sum_{i=1}^{m} \lambda_i f_i(x)$$

The *dual feasibility problem* is

does there exist $\lambda \in \mathbb{R}^m$ such that

$$g(\lambda) < 0$$
$$\lambda \geq 0$$

Theorem of Alternatives

If the dual problem is feasible, then the primal problem is infeasible.

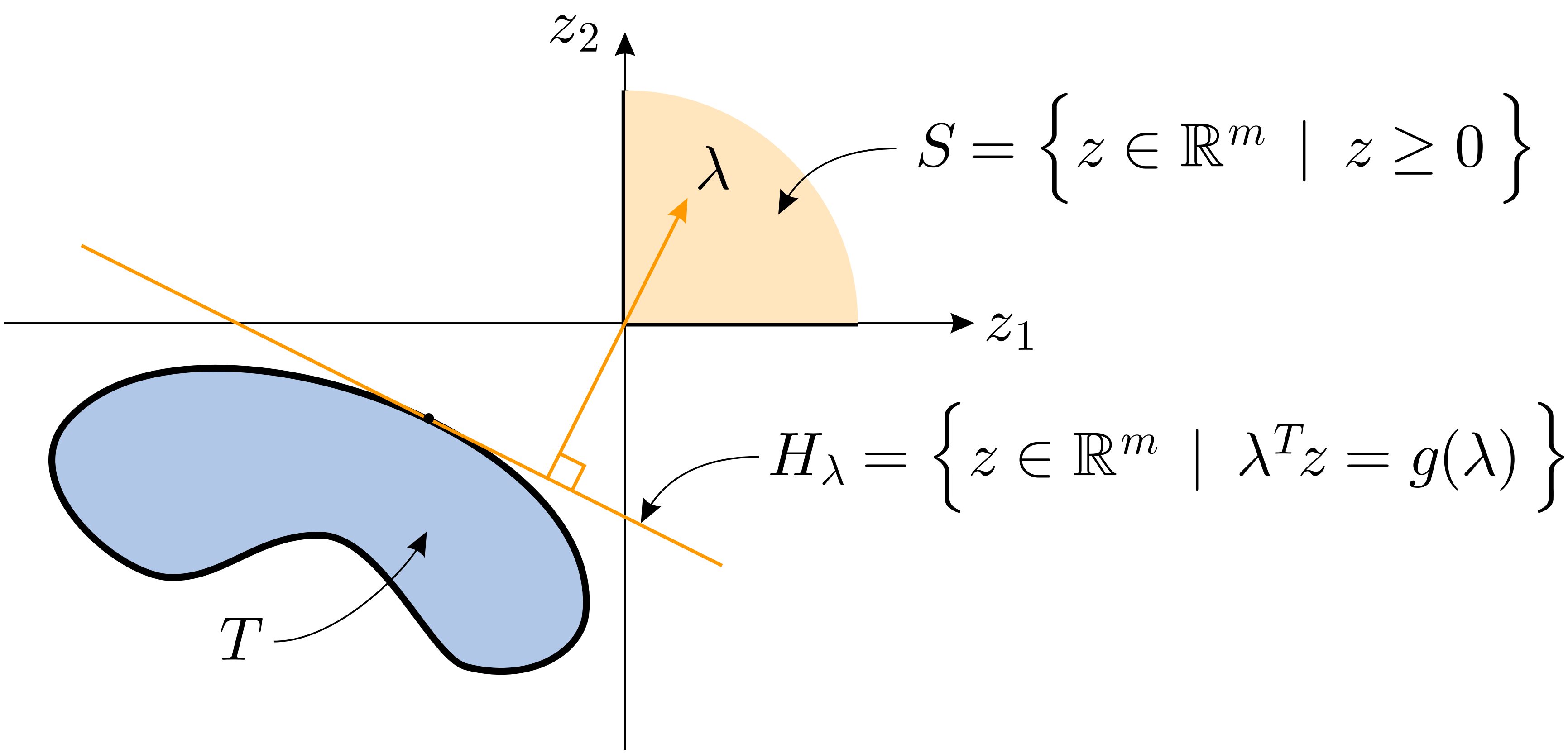
Proof

Suppose the primal problem is feasible, and let $\tilde{x}$ be a feasible point. Then

$$\begin{aligned} g(\lambda) &= \sup_{x \in \mathbb{R}^n} \sum_{i=1}^m \lambda_i f_i(x) \\ &\geq \sum_{i=1}^m \lambda_i f_i(\tilde{x}) \qquad \text{for all } \lambda \in \mathbb{R}^m \end{aligned}$$

and so $g(\lambda) \geq 0$ for all $\lambda \geq 0$.

Geometric Interpretation



if $g(\lambda) < 0$ and $\lambda \geq 0$ then the hyperplane H_λ separates S from T, where

$$T = \left\{ \begin{bmatrix} f_1(x) \\ \vdots \\ f_m(x) \end{bmatrix} \mid x \in \mathbb{R}^n \right\}$$

Certificates

- A dual feasible point gives a *certificate* of infeasibility of the primal problem.
- If the Lagrange dual function g is easy to compute, and we can show $g(\lambda) < 0$, then this is a *proof* that the primal is infeasible.
- One way to do this is to have an explicit expression for

$$g(\lambda) = \sup_x L(x, \lambda)$$

  where for feasibility problems, the Lagrangian is $L(x, \lambda) = \sum_{i=1}^{m} \lambda_i f_i(x)$

- Alternatively, given λ, we may be able to show directly that

$$L(x, \lambda) < -\varepsilon \qquad \text{for all } x \in \mathbb{R}^n$$

  for some $\varepsilon > 0$.