10. The Positivstellensatz

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- Semialgebraic sets
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- Feasibility of semialgebraic sets
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- The real Nullstellensatz
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Basic Semialgebraic Sets

The *basic (closed) semialgebraic set* defined by polynomials $f_1, \ldots, f_m$ is

$$\left\{ x \in \mathbb{R}^n \mid f_i(x) \geq 0 \text{ for all } i = 1, \ldots, m \right\}$$

Examples

- The nonnegative orthant in $\mathbb{R}^n$
- The cone of positive semidefinite matrices
- Feasible set of an SDP; polyhedra and spectrahedra

Properties

- If S_1, S_2 are basic closed semialgebraic sets, then so is $S_1 \cap S_2$; i.e., the class is closed under intersection
- Not closed under union or projection

Semialgebraic Sets

Given the basic semialgebraic sets, we may generate other sets by set theoretic operations; unions, intersections and complements.

A set generated by a finite sequence of these operations on basic semialgebraic sets is called a *semialgebraic set*.

Some examples:

- The set
$$S = \left\{ x \in \mathbb{R}^n \mid f(x) * 0 \right\}$$
is semialgebraic, where $*$ denotes $<, \leq, =, \neq$.
- In particular every real variety is semialgebraic.
- We can also generate the semialgebraic sets via Boolean logical operations applied to polynomial equations and inequalities

Semialgebraic Sets

Every semialgebraic set may be represented as either

- an intersection of unions

$$S = \bigcap_{i=1}^{m} \bigcup_{j=1}^{p_i} \left\{ x \in \mathbb{R}^n \mid \textbf{sign}\, f_{ij}(x) = a_{ij} \right\} \text{ where } a_{ij} \in \{-1, 0, 1\}$$

- a finite union of sets of the form

$$\left\{ x \in \mathbb{R}^n \mid f_i(x) > 0, h_j(x) = 0 \text{ for all } i = 1, \dots, m,\ j = 1, \dots, p \right\}$$

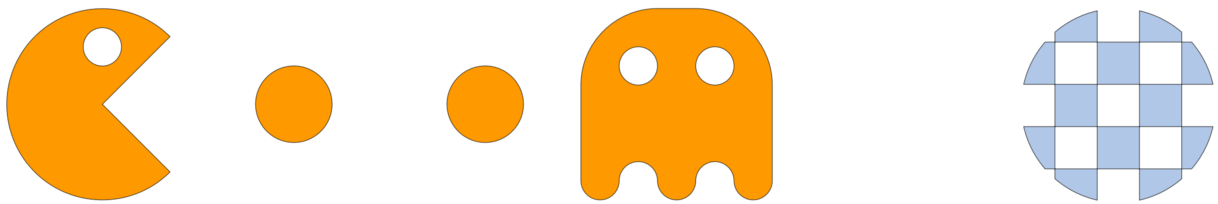
- in $\mathbb{R}$, a finite union of points and open intervals

Every *closed* semialgebraic set is a finite union of basic closed semialgebraic sets; i.e., sets of the form

$$\left\{ x \in \mathbb{R}^n \mid f_i(x) \geq 0 \text{ for all } i = 1, \dots, m \right\}$$

Properties of Semialgebraic Sets

- If S_1, S_2 are semialgebraic, so is $S_1 \cup S_2$ and $S_1 \cap S_2$
- The projection of a semialgebraic set is semialgebraic
- The closure and interior of a semialgebraic sets are both semialgebraic
- Some examples:



Sets that are not Semialgebraic

Some sets are not semialgebraic; for example

- the graph $\{ (x, y) \in \mathbb{R}^2 \mid y = e^x \}$
- the infinite staircase $\{ (x, y) \in \mathbb{R}^2 \mid y = \lfloor x \rfloor \}$
- the infinite grid $\mathbb{Z}^n$

Tarski-Seidenberg and Quantifier Elimination

Tarski-Seidenberg theorem: if $S \subset \mathbb{R}^{n+p}$ is semialgebraic, then so are

- $\{\, x \in \mathbb{R}^n \mid \exists\, y \in \mathbb{R}^p \; (x, y) \in S \,\}$ (closure under projection)
- $\{\, x \in \mathbb{R}^n \mid \forall\, y \in \mathbb{R}^p \; (x, y) \in S \,\}$ (complements and projections)

i.e., quantifiers do not add any expressive power

Cylindrical algebraic decomposition (CAD) may be used to compute the semialgebraic set resulting from quantifier elimination

Feasibility of Semialgebraic Sets

Suppose S is a semialgebraic set; we'd like to solve the feasibility problem

Is S non-empty?

More specifically, suppose we have a semialgebraic set represented by polynomial inequalities and equations

$$S = \left\{ x \in \mathbb{R}^n \mid f_i(x) \geq 0,\ h_j(x) = 0 \text{ for all } i = 1, \dots, m,\ j = 1, \dots, p \right\}$$

- Important, non-trivial result: the feasibility problem is *decidable*.
- But NP-hard (even for a single polynomial, as we have seen)
- We would like to *certify* infeasibility

Certificates So Far

- *The Nullstellensatz:* a necessary and sufficient condition for feasibility of *complex* varieties

$$\left\{ x \in \mathbb{C}^n \mid h_i(x) = 0 \; \forall\, i \right\} = \emptyset \qquad \Longleftrightarrow \qquad -1 \in \textbf{ideal}\{h_1, \ldots, h_m\}$$

- *Valid inequalities:* a *sufficient* condition for infeasibility of *real basic* semialgebraic sets

$$\left\{ x \in \mathbb{R}^n \mid f_i(x) \geq 0 \; \forall\, i \right\} = \emptyset \qquad \Longleftarrow \qquad -1 \in \textbf{cone}\{f_1, \ldots, f_m\}$$

- *Linear Programming:* necessary and sufficient conditions via duality for *real linear* equations and inequalities

Certificates So Far

Degree \ Field	Complex	Real
Linear	*Range/Kernel* Linear Algebra	*Farkas Lemma* Linear Programming
Polynomial	*Nullstellensatz* Bounded degree: LP Groebner bases	*????* ????

We'd like a method to construct certificates for

- *polynomial* equations
- over the *real* field

Real Fields and Inequalities

If we can test feasibility of *real* equations then we can also test feasibility of real *inequalities* and *inequations*, because

- *inequalities:* there exists $x \in \mathbb{R}$ such that $f(x) \geq 0$ if and only if

  $$\text{there exists } (x, y) \in \mathbb{R}^2 \text{ such that } f(x) = y^2$$

- *strict inequalities:* there exists x such that $f(x) > 0$ if and only if

  $$\text{there exists } (x, y) \in \mathbb{R}^2 \text{ such that } y^2 f(x) = 1$$

- *inequations:* there exists x such that $f(x) \neq 0$ if and only if

  $$\text{there exists } (x, y) \in \mathbb{R}^2 \text{ such that } y f(x) = 1$$

The underlying theory for real polynomials called *real algebraic geometry*

Real Varieties

The *real variety* defined by polynomials $h_1, \dots, h_m \in \mathbb{R}[x_1, \dots, x_n]$ is

$$\mathcal{V}_{\mathbb{R}}\{h_1, \dots, h_m\} = \left\{ x \in \mathbb{R}^n \mid h_i(x) = 0 \text{ for all } i = 1, \dots, m \right\}$$

We'd like to solve the feasibility problem; is $\mathcal{V}_{\mathbb{R}}\{h_1, \dots, h_m\} \neq \emptyset$?

We know

- Every polynomial in $\mathbf{ideal}\{h_1, \dots, h_m\}$ vanishes on the feasible set.
- The (complex) Nullstellensatz:

$$-1 \in \mathbf{ideal}\{h_1, \dots, h_m\} \qquad \Longrightarrow \qquad \mathcal{V}_{\mathbb{R}}\{h_1, \dots, h_m\} = \emptyset$$

- But this condition is not necessary over the reals

The Real Nullstellensatz

Recall Σ is the cone of polynomials representable as *sums of squares*.

Suppose $h_1, \ldots, h_m \in \mathbb{R}[x_1, \ldots, x_n]$.

$$-1 \in \Sigma + \mathbf{ideal}\{h_1, \ldots, h_m\} \qquad \iff \qquad \mathcal{V}_{\mathbb{R}}\{h_1, \ldots, h_m\} = \emptyset$$

Equivalently, there is no $x \in \mathbb{R}^n$ such that

$$h_i(x) = 0 \qquad \text{for all } i = 1, \ldots, m$$

if and only if there exists $t_1, \ldots, t_m \in \mathbb{R}[x_1, \ldots, x_n]$ and $s \in \Sigma$ such that

$$-1 = s + t_1 h_1 + \cdots + t_m h_m$$

Example

Suppose $h(x) = x^2 + 1$. Then clearly $\mathcal{V}_{\mathbb{R}}\{h\} = \emptyset$

We saw earlier that the complex Nullstellensatz cannot be used to prove emptyness of $\mathcal{V}_{\mathbb{R}}\{h\}$

But we have

$$-1 = s + th$$

with

$$s(x) = x^2 \qquad \text{and} \qquad t(x) = -1$$

and so the real Nullstellensatz implies $\mathcal{V}_{\mathbb{R}}\{h\} = \emptyset$.

The polynomial equation $-1 = s + th$ gives a certificate of infeasibility.

The Positivstellensatz

We now turn to feasibility for *basic semialgebraic sets*, with primal problem

Does there exist $x \in \mathbb{R}^n$ such that

$$f_i(x) \geq 0 \qquad \text{for all } i = 1, \ldots, m$$
$$h_j(x) = 0 \qquad \text{for all } j = 1, \ldots, p$$

Call the feasible set S; recall

- every polynomial in $\mathbf{cone}\{f_1, \ldots, f_m\}$ is nonnegative on S
- every polynomial in $\mathbf{ideal}\{h_1, \ldots, h_p\}$ is zero on S

The *Positivstellensatz* (Stengle 1974)

$$S = \emptyset \quad \Longleftrightarrow \quad -1 \in \mathbf{cone}\{f_1, \ldots, f_m\} + \mathbf{ideal}\{h_1, \ldots, h_m\}$$

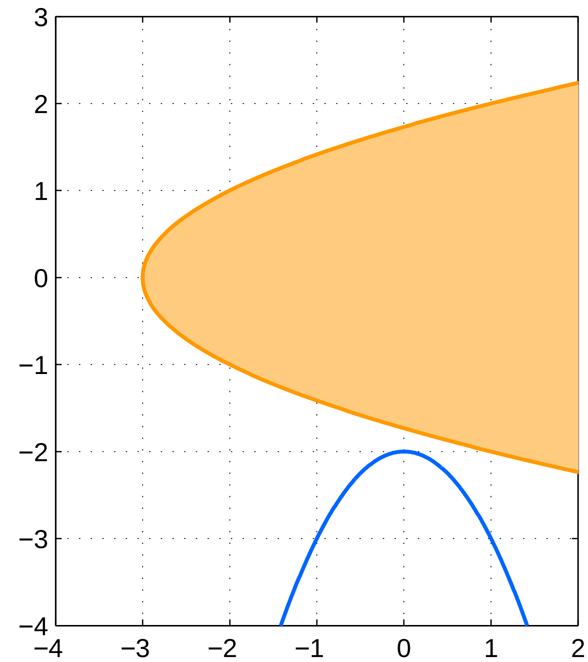
Example

Consider the feasibility problem

$$S = \{ (x, y) \in \mathbb{R}^2 \mid f(x, y) \geq 0, h(x, y) = 0 \}$$

where

$$f(x, y) = x - y^2 + 3$$
$$h(x, y) = y + x^2 + 2$$



By the P-satz, the primal is infeasible if and only if there exist polynomials $s_1, s_2 \in \Sigma$ and $t \in \mathbb{R}[x, y]$ such that

$$-1 = s_1 + s_2 f + th$$

A certificate is given by

$$s_1 = \frac{1}{3} + 2\left(y + \frac{3}{2}\right)^2 + 6\left(x - \frac{1}{6}\right)^2, \quad s_2 = 2, \quad t = -6.$$

Explicit Formulation of the Positivstellensatz

The primal problem is

Does there exist $x \in \mathbb{R}^n$ such that

$$f_i(x) \geq 0 \qquad \text{for all } i = 1, \ldots, m$$
$$h_j(x) = 0 \qquad \text{for all } j = 1, \ldots, p$$

The dual problem is

Do there exist $t_i \in \mathbb{R}[x_1, \ldots, x_n]$ and $s_i, r_{ij}, \ldots \in \Sigma$ such that

$$-1 = \sum_i h_i t_i + s_0 + \sum_i s_i f_i + \sum_{i \neq j} r_{ij} f_i f_j + \cdots$$

These are *strong alternatives*

Testing the Positivstellensatz

Do there exist $t_i \in \mathbb{R}[x_1, \ldots, x_n]$ and $s_i, r_{ij}, \ldots \in \Sigma$ such that

$$-1 = \sum_i t_i h_i + s_0 + \sum_i s_i f_i + \sum_{i \neq j} r_{ij} f_i f_j + \cdots$$

- This is a convex feasibility problem in $t_i, s_i, r_{ij}, \ldots$
- To solve it, we need to choose a subset of the cone to search; i.e., the maximum degree of the above polynomial; then the problem is a *semidefinite program*
- This gives a *hierarchy* of syntactically verifiable certificates
- The validity of a certificate may be easily checked; e.g., linear algebra, random sampling
- Unless NP=co-NP, the certificates cannot *always* be polynomially sized.

Example: Farkas Lemma

The primal problem; does there exist $x \in \mathbb{R}^n$ such that

$$Ax + b \geq 0 \qquad Cx + d = 0$$

Let $f_i(x) = a_i^T x + b_i$, $h_i(x) = c_i^T x + d_i$. Then this system is infeasible if and only if

$$-1 \in \mathbf{cone}\{f_1, \dots, f_m\} + \mathbf{ideal}\{h_1, \dots, h_p\}$$

Searching over *linear combinations*, the primal is infeasible if there exist $\lambda \geq 0$ and μ such that

$$\lambda^T(Ax + b) + \mu^T(Cx + d) = -1$$

Equating coefficients, this is equivalent to

$$\lambda^T A + \mu^T C = 0 \quad \lambda^T b + \mu^T d = -1 \quad \lambda \geq 0$$

Hierarchy of Certificates

- Interesting connections with logic, proof systems, etc.
- Failure to prove infeasibility (may) provide points in the set.
- Tons of applications:
  optimization, copositivity, dynamical systems, quantum mechanics...

Special Cases

Many known methods can be interpreted as fragments of P-satz refutations.

- LP duality: linear inequalities, constant multipliers.
- S-procedure: quadratic inequalities, constant multipliers
- Standard SDP relaxations for QP.
- The *linear representations* approach for functions f strictly positive on the set defined by $f_i(x) \geq 0$.

$$f(x) = s_0 + s_1 f_1 + \cdots + s_n f_n, \qquad s_i \in \Sigma$$

Converse Results

- *Losslessness:* when can we restrict *a priori* the class of certificates?
- Some cases are known; e.g., additional conditions such as linearity, perfect graphs, compactness, finite dimensionality, etc, can ensure specific *a priori* properties.

Example: Boolean Minimization

$$x^T Q x \leq \gamma$$
$$x_i^2 - 1 = 0$$

A P-satz refutation holds if there is $S \succeq 0$ and $\lambda \in \mathbb{R}^n$, $\varepsilon > 0$ such that

$$-\varepsilon = x^T S x + \gamma - x^T Q x + \sum_{i=1}^{n} \lambda_i (x_i^2 - 1)$$

which holds if and only if there exists a diagonal Λ such that $Q \succeq \Lambda$, $\gamma = \mathbf{trace}\, \Lambda - \varepsilon$.

The corresponding optimization problem is

$$\begin{array}{ll} \text{maximize} & \mathbf{trace}\, \Lambda \\ \text{subject to} & Q \succeq \Lambda \\ & \Lambda \text{ is diagonal} \end{array}$$

Example: S-Procedure

The primal problem; does there exist $x \in \mathbb{R}^n$ such that

$$x^T F_1 x \geq 0$$
$$x^T F_2 x \geq 0$$
$$x^T x = 1$$

We have a P-satz refutation if there exists $\lambda_1, \lambda_2 \geq 0$, $\mu \in \mathbb{R}$ and $S \succeq 0$ such that

$$-1 = x^T S x + \lambda_1 x^T F_1 x + \lambda_2 x^T F_2 x + \mu(1 - x^T x)$$

which holds if and only if there exist $\lambda_1, \lambda_2 \geq 0$ such that

$$\lambda_1 F_1 + \lambda_2 F_2 \leq -I$$

Subject to an additional mild constraint qualification, this condition is also *necessary* for infeasibility.

Exploiting Structure

What algebraic properties of the polynomial system yield efficient computation?

- *Sparseness:* few nonzero coefficients.
    - Newton polytopes techniques
    - Complexity does not depend on the degree
- *Symmetries:* invariance under a transformation group
    - Frequent in practice. Enabling factor in applications.
    - Can reflect underlying physical symmetries, or modelling choices.
    - SOS on *invariant rings*
    - Representation theory and invariant-theoretic techniques.
- *Ideal structure:* Equality constraints.
    - SOS on *quotient rings*
    - Compute in the coordinate ring. Quotient bases (Groebner)