Semidefinite Programming Relaxations and Algebraic Optimization in Control

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IEEE Conference on Decision and Control
December 8, 2003

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Schedule

Topics

- Convexity and duality
- Algebra and duality
- Applications
- Computation
- Complexity

Guest Speakers

- *Lieven Vandenberghe, UCLA* Interior-point methods
- *Stephen Prajna, Caltech* Applications

Overview

Mathematical and computational theory, and applications to combinatorial, non-convex and nonlinear problems

- Semidefinite programming
- Real algebraic geometry
- Duality and certificates

Schedule

- Introduction
- Convexity and Duality
- Quadratically Constrained Quadratic Programming
- Algebra and Duality
- Linear Inequalities and Elimination
- Complexity
- The Algebraic-Geometric Dictionary
- Sums of Squares
- Interpretations, Liftings, SOS and Moments
- The Positivstellensatz
- Applications
- Semialgebraic Liftings
- Further Applications
- Interior-Point Methods
- Summary

Optimization Problems

A familiar problem

$$\begin{array}{rll} \text{minimize} & f_0(x) & \\ \text{subject to} & f_i(x) \leq 0 & \text{for all } i = 1, \ldots, m \\ & h_i(x) = 0 & \text{for all } i = 1, \ldots, p \end{array}$$

- $x \in \mathbb{R}^n$ is the variable
- $f_0 : \mathbb{R}^n \to \mathbb{R}$ is the *objective function*
- $f_i : \mathbb{R}^n \to \mathbb{R}$ for $i = 1, \ldots, m$ define *inequality constraints*
- $h_i : \mathbb{R}^n \to \mathbb{R}$ for $i = 1, \ldots, p$ define *equality constraints*

Discrete Problems: LQR with Binary Inputs

- linear discrete-time system $x(t+1) = Ax(t) + Bu(t)$ on interval $t = 0, \ldots, N$

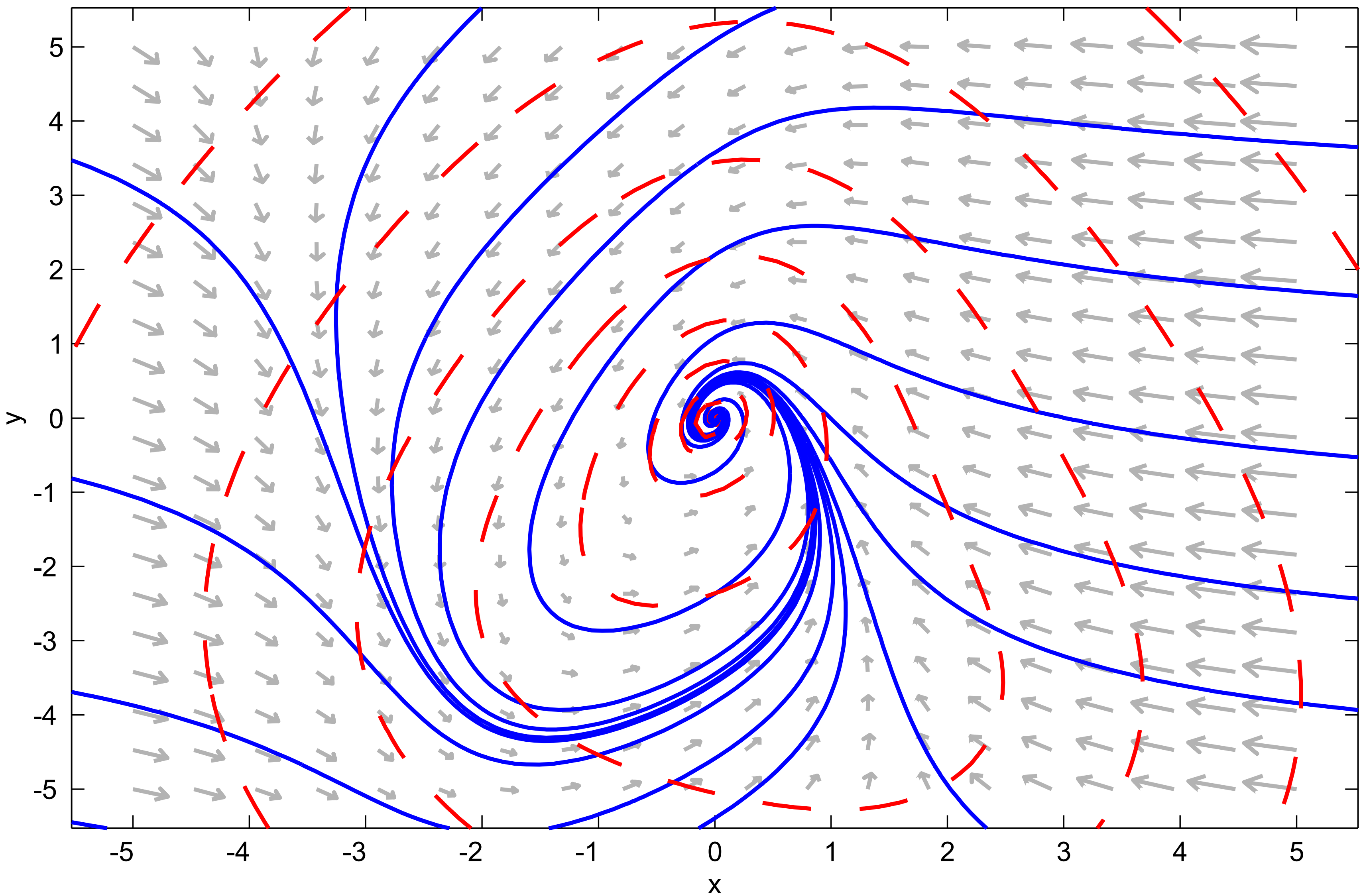
- objective is to minimize the quadratic tracking error

$$\sum_{t=0}^{N-1} \big(x(t) - r(t)\big)^T Q \big(x(t) - r(t)\big)$$

- using binary inputs

$$u_i(t) \in \{-1, 1\} \qquad \text{for all } i = 1, \ldots, m, \text{ and } t = 0, \ldots, N-1$$

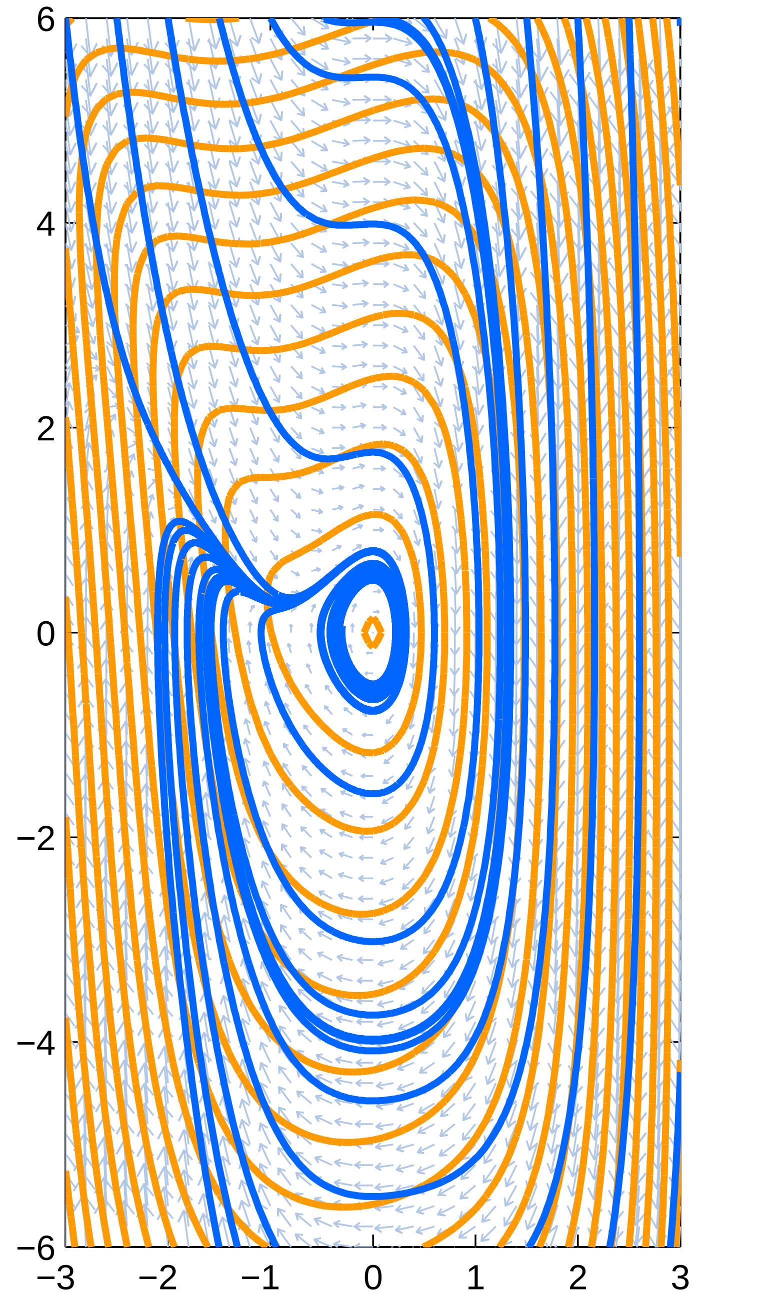
Nonlinear Problems: Lyapunov Stability



Lyapunov Functions

$$\dot{x} = y$$

$$\dot{y} = -4\,x^3 - 2\,x^2 y - \frac{15}{2}\,x^2 - 4\,x$$



- Nonlinear mass-spring system
- Sublevel sets of Lyapunov function are not convex

Entanglement and Quantum Mechanics

- Entanglement is a behavior of quantum states, which cannot be explained classically.
- Responsible for many of the non-intuitive properties, and computational power of quantum devices.

A bipartite mixed quantum state ρ is *separable* (not *entangled*) if

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i| \otimes |\phi_i\rangle\langle\phi_i| \qquad \sum p_i = 1$$

for some ψ_i, ϕ_i.

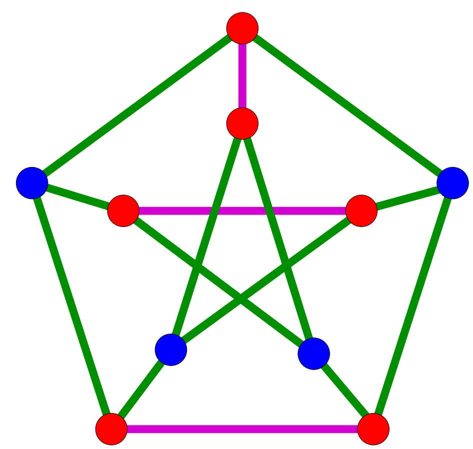
Given ρ, how to *decide* and *certify* if it is entangled?

Graph problems

Graph problems appear in many areas: MAX-CUT, independent set, cliques, etc.

MAX CUT partitioning

- Partition the nodes of a graph in two disjoint sets, maximizing the number of edges between sets.



- Practical applications (circuit layout, etc.)
- NP-complete.

How to compute bounds, or exact solutions, for this kind of problems?

Facility Location

- Given a set of n *cities*
- We'd like to open at most m facilities
- And assign each city to exactly one facility