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Complexity of stability and controllability of elementary hybrid systems[1]

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We consider classes of nonlinear systems that include simple hybrid systems and prove that questions related to the stability and controllability of these systems are either undecidable or computationally intractable.

Abstract

In this paper, we consider simple classes of nonlinear systems and prove that basic questions related to their stability and controllability are either undecidable or computationally intractable (NP-hard). As a special case, we consider a class of hybrid systems in which the state space is partitioned into two halfspaces, and the dynamics in each halfspace correspond to a different linear system. © 1999 Elsevier Science Ltd. All rights reserved.

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1. Introduction

In recent years, much research has focused on *hybrid* systems. These are systems that involve a combination of continuous dynamics (e.g., differential equations or linear evolution equations) and discrete dynamics. The motivation lies in the fact that most complex systems involve a physical layer described by continuous variables, together with higher level layers involving symbolic manipulations and discrete supervisory decisions. Applications range from intelligent traffic systems to industrial process control.

Hybrid systems can be usually described by state space models, using a suitably defined state space (often the Cartesian product of continuous and discrete sets). Classical systems theory provides us with efficient methods for analyzing and controlling certain classes of continuous-variable systems (e.g., linear systems) and certain classes of discrete-variable systems (e.g., finite state Markov chains). However, equally efficient generalizations are not available even for the simplest classes of hybrid systems. This is thought to be a reflection of the inherent complexity of such systems. The research reported in this paper is aimed at elucidating this complexity.

We provide two types of complexity results. Some of the problems presented are shown *undecidable*, that is, they are not amenable to an algorithmic solution. Other problems are shown *NP-hard*, meaning that although these problems may be algorithmically solvable, no *efficient* (polynomial time) algorithm is possible, assuming the validity of a long-standing conjecture in computer science (P $\neq$ NP), which is widely believed to be true. In practice, an NP-hardness result is interpreted to mean that a search for a polynomial time algorithm should be abandoned: if such an algorithm were found, it would immediately imply the falsity of the P $\neq$ NP conjecture, and would lead to efficient algorithms for many other

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longstanding problems for which no such algorithm has been found so far. In order to facilitate readers who are unfamiliar with the notions of undecidability and NP-hardness, we provide a brief discussion below. (Familar readers may skip the next four paragraphs.) We also refer to Garey and Johnson (1979), Hopcroft and Ullman (1969) and Papadimitriou (1994) for rigorous definitions and proofs. Surveys of decidability and complexity results presently available for control problems appear in Blondel and Tsitsiklis (1998, 1999).

We only look at problems that are formulated as *decision problems* (problems with "yes/no" answers), where we are asked to decide whether a given instance of the problem under consideration has a certain property. For example, the problem of deciding whether a given real matrix is stable is a decision problem but the problem of finding its spectral radius is not. A solution for a given problem must be in the form of an *algorithm* that takes an instance as an input and is guaranteed to terminate with the correct answer. A problem is called *decidable* if such an algorithm exists, and *undecidable* otherwise. What constitutes an acceptable algorithm may depend on an underlying model of computation. Various models of (digital) computation are available, but reasonable models have turned out to be equivalent, in the sense that they all lead to the same set of decidable problems. Thus, decidability is a well-defined, mathematically sound, and machine independent concept.

We now turn to the notions of running time and NP-hardness. We consider the running time (number of steps) of an algorithm for a given problem, in the worst case over all instances of a fixed *size*. (The size of an instance is defined as the number of bits needed to describe that instance according to some prespecified format.) If this running time increases no faster than some polynomial function of the size, we say that the algorithm runs in *polynomial time*. A problem that admits a polynomial time algorithm is said to belong to the class P, and is considered to be efficiently solvable. Once more, it turns out that the definition of the class P is highly robust, and the class P remains the same for different reasonable models of computation. In order to show that a problem cannot be solved efficiently, one would like to prove that it does *not* belong to P. Such results are hard to establish and computer scientists rely on a different approach for showing that a problem is (likely to be) difficult. There is a class NP (which stands for *Nondeterministic Polynomial time*) that includes all of P, but also contains a large number of problems for which no polynomial time algorithm has yet been found (integer programming is one such problem). It is not known whether P = NP, but it is generally believed that this is not the case.

Consider two different problems, say problems A and B, and suppose that there exists a polynomial time *reduction* of problem B to problem A. (By this, we mean an algorithm that takes an instance of B as input, runs for polynomial time, and produces an equivalent instance of problem A, i.e., with the same "yes" or "no" answer.) If problem A admits a polynomial time algorithm, it can be combined with the reduction of B to A, to obtain a polynomial time algorithm for problem B. In that sense, problem B is no harder than problem A or, conversely, problem A is at least as hard as problem B. We say that a problem in NP is NP-complete if every problem in NP can be reduced to it. In that sense, an NP-complete problem is a "hardest" problem in NP. Many problems (including integer programming, for example) are known to be NP-complete. More generally, we say that a problem is NP-hard if every problem in NP can be reduced to it (this is the same as NP-completeness, without the requirement that the problem belongs to NP). Note that a polynomial time algorithm for some NP-hard problem would again translate to polynomial time algorithms for all problems in NP. Once a problem is shown to be NP-hard, this does not prove that no efficient algorithm exists for that problem. But it shows that obtaining such an algorithm is as hard as establishing that P = NP, which is neither easy nor likely to be true.

The proof technique for showing that a problem A is NP-hard makes use of reductions. However, instead of showing that every problem in NP can be reduced to A, it suffices to reduce a single NP-complete problem B to A. There are thousands of problems that are known to be NP-complete and can play the role of B in the above schema; typically, one looks for such a problem that bears some relation with the problem A of interest.

We now provide some insights on the complexity of problems involving hybrid systems, and illustrate some of the above concepts. Consider a system with state $(x_t, q_t) \in \mathbf{R}^n \times \{1, \ldots, m\}$ where x_t and q_t are, respectively, the continuous and discrete parts of the state. Let A_i $(i = 1, \ldots, m)$ be square matrices and let the dynamics of x_t depend on the discrete state as follows:

$$x_{t+1} = A_i x_t \quad \text{when } q_t = i.$$

In addition, let a finite partition of $\mathbf{R}^n$ be given, $\mathbf{R}^n = H_1 \cup H_2 \cup \cdots \cup H_m$, and suppose that the discrete state q_t depends only on the location of the continuous state x_t in the partition, i.e.,

$$q_t = i \quad \text{when } x_t \in H_i.$$

Then, the overall hybrid system takes the form of a nonlinear system

$$x_{t+1} = A_i x_t \quad \text{when } x_t \in H_i. \tag{1}$$

If the partition consists of two regions separated by a hyperplane, the system becomes

$$x_{t+1} = \begin{cases} A_1 x_t & \text{when } c^{\mathrm{T}} x_t \geq 0, \\ A_2 x_t & \text{when } c^{\mathrm{T}} x_t < 0. \end{cases} \tag{2}$$

A system is *stable* if its state vector always converges to zero. Deciding stability for hybrid systems as simple as Eq. (2) is already a nontrivial task, as we now explain using a simple example. We build a state space model for a system described by a state vector (v_t, y_t, z_t), where v_t and y_t are scalars and z_t is a vector in $\mathbf{R}^n$. The dynamics of the system are of the form

$$\begin{pmatrix} v_{t+1} \\ y_{t+1} \\ z_{t+1} \end{pmatrix} = \begin{pmatrix} 1/4 & 0 & 0 \\ -1/4 & 1/2 & 0 \\ 0 & 0 & A_+ \end{pmatrix} \begin{pmatrix} v_t \\ y_t \\ z_t \end{pmatrix} \quad \text{when } y_t \geq 0$$

and

$$\begin{pmatrix} v_{t+1} \\ y_{t+1} \\ z_{t+1} \end{pmatrix} = \begin{pmatrix} 1/4 & 0 & 0 \\ 1/4 & 1/2 & 0 \\ 0 & 0 & A_- \end{pmatrix} \begin{pmatrix} v_t \\ y_t \\ z_t \end{pmatrix} \quad \text{when } y_t < 0$$

This hybrid system consists of two linear systems, each of which is enabled in one of the two halfspaces, as determined by the sign of y_t.

Let us now look at the evolution of an initial state vector (v_0, y_0, z_0). Suppose that $v_0 = 1$ in which case we have $v_t = 4^{-t}$ for all t. Suppose in addition, that y_0 can take any value in $[-1, 1]$. Then, it is easily seen that y_1 can take any value in $[-\frac{1}{4}, \frac{1}{4}]$, no matter what was the sign of y_0. Continuing inductively, we see that y_t can take any value in $[-4^{-t}, 4^{-t}]$, can have either sign, and this is independent of the signs of y_s for $s < t$. This shows that every possible sign sequence can be generated by suitable choice of y_0. Hence, the dynamics of the state subvector z_t are of the form $z_{t+1} = A_t z_t$, where each A_t is an arbitrary element of $\{A_-, A_+\}$. We conclude that the state vector converges to zero, for all possible initial states, if and only if all sequences of products of the matrices A_- and A_+ (taken in an arbitrary order) converge to zero.

Unfortunately, a test for the stability of all possible sequences of products of two matrices is not available. The decidability of this problem is a major open question and is intimately related to the so-called "finiteness conjecture" (see, e.g., Daubechies and Lagarias (1992), Lagarias and Wang (1995); Gurvits (1995, 1997)). If the stability of all possible sequences of products of two matrices turns out to be undecidable, it will immediately follow that the stability of the class of hybrid systems of the form (2) is also undecidable. Given the present state of knowledge, we are unable to prove such an undecidability result. On the other hand, NP-hardness of the stability problem for systems of the form (2) is obtained with a straightforward adaptation of the arguments in Tsitsiklis and Blondel (1997).

In Section 2, we build on this last observation and prove NP-hardness of the stability problem for many more classes of systems. Let us note that systems of the form (2) can also be written in the form

$$x_{t+1} = (B_0 + v(c^{\mathrm{T}} x_t) B_1) x_t \tag{3}$$

with $B_0 = A_1$, $B_1 = A_2 - A_1$, and with the function v defined by $v(\alpha) = 0$ for $\alpha \geq 0$, and by $v(\alpha) = 1$ for $\alpha < 0$. In Theorem 1, we consider nonlinear systems of the form (3) where v is an arbitrary scalar function. We show that for a large class of nonconstant functions v, the stability of these systems is NP-hard to decide. As a special case, our result applies to the particular function v defined above, and so the stability of systems of the form (2) is NP-hard to decide.

In Section 3, we consider classes of elementary hybrid systems similar to Eq. (2), but also involving a control variable. The nth-dimensional *sign system* associated with $A_+, A_0, A_- \in \mathbf{R}^{n \times n}$ and $b, c \in \mathbf{R}^n$ is the system

$$x_{t+1} = A_{\mathrm{sgn}(c^{\mathrm{T}} x_t)} x_t + b u_t, \quad t = 0, 1, \ldots,$$

where $\mathrm{sgn}(\cdot)$ is the sign function defined by

$$\mathrm{sgn}(x) = \begin{cases} + & \text{when } x > 0, \\ 0 & \text{when } x = 0, \\ - & \text{when } x < 0. \end{cases}$$

In Theorem 2, we establish that null-controllability and complete reachability are both undecidable for such systems. A related result is given by Toker (1996) who considers a class of systems similar to sign systems. He shows that the question of deciding whether all possible control actions drive a given initial state to the origin is undecidable. Our problem is different in that we do not consider a single given initial state, and in that we ask whether some, not all, control laws drive the state to the origin. Theorem 2 is also related to our earlier work on the complexity of certain questions involving products of matrices coming from a given finite family (Blondel and Tsitsiklis (1997); Tsitsiklis and Blondel (1997)). In our earlier work, matrices could be multiplied in an arbitrary order. The present work is different in that the choice of the next matrix in the product is determined by a feedback mechanism involving the state of the system.

Systems of the form (1) are the *piecewise linear* systems introduced by Sontag (1981), and for which some complexity results are already available; see Sontag (1995) for a survey as well as for results for other types of nonlinear systems. The systems (1) are also similar to the *piecewise constant derivative* systems analyzed by Asarin, Maler, and Pnueli. A piecewise constant derivative (PCD) system is given by a finite partition of $\mathbf{R}^n$, $\mathbf{R}^n = H_1 \cup H_2 \cup \cdots \cup H_m$, and by slope vectors b_i for every region H_i of the partition. We assume that each region H_i is a polyhedral set. On any given region of the partition, the state $x(t)$ has a constant derivative,

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = b_i \quad \text{when } x(t) \in H_i.$$

Then, the trajectories $x(t)$ are continuous broken lines, with breaking points occurring on the boundaries of the regions. Asarin et al. (1995) provide some results on point-to-point reachability for such systems. In particular, for given states x_b and x_e, the problem of deciding whether x_e is reached by a trajectory starting from x_b, is decidable for systems of dimension two, but is undecidable for systems of dimension three or more. This undecidability result is obtained by showing that PCD systems can simulate Turing machines. By using a universal Turing machine, undecidability of point-to-point reachability can be obtained for a *particular* PCD system in dimension three. Considering this particular system, it is then easy to construct a partition of $\mathbf{R}^4$ into finitely many polyhedral sets H_i, and 4×4 matrices A_i for every region H_i, such that the problem of determining, for given $x_b, x_e \in \mathbf{R}^4$, whether x_b reaches x_e when

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = A_i x \quad \text{when } x \in H_i$$

is undecidable. Thus, point-to-point reachability for continuous time systems analogous to those in Eq. (1) is undecidable.

Turing machine simulations are possible by other types of dynamical systems; see, for example, Bournez and Cosnard (1995) for simulation by analog automata, Siegelmann and Sontag (1995) for simulation by linear systems with saturation nonlinearities, Branicky (1995) for simulation by differential equations, and Henzinger et al. (1995) for simulation by timed automata.

In all of these constructions, the regions of the partition are used to encode the states of a Turing machine and this usually leads to a high number of regions. A novel aspect of our results, when compared with those mentioned above, is that they demonstrate undecidability for hybrid systems with very few regions.

2. Autonomous systems

A *discrete-time autonomous system* $f: \mathbf{R}^n \mapsto \mathbf{R}^n$ is said to be *globally asymptotically stable*[2] (or, for short, *asymptotically stable*) if the sequences defined by

$$x_{t+1} = f(x_t), \quad t = 0, 1, \ldots,$$

converge to the origin for all initial states $x_0 \in \mathbf{R}^n$.

Let A be an $n \times n$ real matrix. It is well known that the linear system $x_{t+1} = Ax_t$ is asymptotically stable if and only if all eigenvalues of A have magnitude strictly less than one. Furthermore, asymptotic stability can be decided efficiently, e.g., by solving a Lyapunov equation. No such simple and computationally efficient test exists for general nonlinear systems.

In this section, we define particular classes of systems involving a single scalar nonlinearity, and we prove that algorithms for deciding asymptotic stability of systems in any one of our classes are inherently inefficient. Unless $P = NP$, the running time of any such algorithm must increase faster than any polynomial in the size of the description of the system. Some of our classes are elementary and can be viewed as the "least nonlinear" systems. In particular, one of our classes corresponds to systems that are linear on each side of a hyperplane.

Systems with a single scalar nonlinearity. Let us fix a scalar function $v: \mathbf{R} \mapsto \mathbf{R}$. The v-system associated with $n \geq 1$, $A_0, A_1 \in \mathbf{R}^{n \times n}$, and $c \in \mathbf{R}^n$, is defined by

$$x_{t+1} = (A_0 + v(c^{\mathrm{T}} x_t) A_1) x_t, \quad t = 0, 1, \ldots .$$

(Here, the superscript T denotes matrix transposition.) When v is a constant function, v-systems are linear and their stability can be decided easily. We show in Theorem 1 below that for a broad variety of nonconstant functions v, the stability of v-systems is NP-hard to decide.

Let us note that stability can be difficult to check for the simple reason that v may be difficult to compute. For this reason, the result that we present below is of interest primarily for the case where v is an easily computable function.

Theorem 1. *Let us fix a nonconstant scalar function $v: \mathbf{R} \mapsto \mathbf{R}$ such that*

$$\lim_{x \to -\infty} v(x) \leq v(x) \leq \lim_{x \to +\infty} v(x)$$

for all $x \in \mathbf{R}$, and where the limits are assumed to exist. Then, the asymptotic stability of v-systems is NP-hard to decide.

Proof. Our proof relies on a construction developed in Tsitsiklis and Blondel (1997), which in turn is based on a reduction technique introduced in Papadimitriou and Tsitsiklis (1987). Rather than repeating here the construction in Tsitsiklis and Blondel (1997), we simply state its conclusions, in the form of the lemma that follows. The lemma makes reference to 3SAT, which is the Boolean satisfiability problem with three literals per clause, and is a well-known NP-complete problem. For a precise definition of 3SAT, see Garey and Johnson (1979).

Lemma 1. *Given an instance of 3SAT with n variables and m clauses, we can construct (in polynomial time) two matrices A_0 and A_1, of dimensions $r \times r$, where $r = (n+1)(m+1)$, whose entries belong to $\{0,1\}$, and with the following properties*:

(a) *If we have a "yes" instance of 3SAT, there exist indices $k_1, k_2, \ldots, k_{n+2} \in \{0,1\}$, and a nonnegative nonzero integer vector x such that $A_{k_{n+2}} \cdots A_{k_2} A_{k_1} x = mx$.*

(b) *If we have a "no" instance of 3SAT, then $\|A_{k_{n+2}} \cdots A_{k_2} A_{k_1} x\| \leq (m-1)\|x\|$, for every vector x, and for every choice of indices $k_1, k_2, \ldots, k_{n+2} \in \{0,1\}$. Here, and throughout the paper, $\|\cdot\|$ stands for the maximum (ℓ_∞) norm.*

[2] Note that our definitions of stability are somewhat different from the commonly used ones.

Let us now fix a nonconstant function $v(\cdot)$ with

$$\lim_{x\to-\infty} v(x) \le v(x) \le \lim_{x\to+\infty} v(x),$$

for all $x\in\mathbf{R}$, and let $a_- = \lim_{x\to-\infty} v(x)$ and $a_+ = \lim_{x\to+\infty} v(x)$. For simplicity and ease of exposition, we assume that a_- and a_+ are rational numbers. This restriction is not essential and can be easily removed, as discussed at the end of the proof.

Since we have assumed that $v(\cdot)$ is not constant, we have $a_- < a_+$. Given an instance of 3SAT, we construct the matrices A_0 and A_1 as in Lemma 1. We then let

$$B_0 = \frac{a_+A_0 - a_-A_1}{a_+ - a_-}, \qquad B_1 = \frac{A_1 - A_0}{a_+ - a_-}.$$

It is seen that for any $a\in\mathbf{R}$, we have

$$B_0 + aB_1 = \frac{a_+ - a}{a_+ - a_-}A_0 + \frac{a - a_-}{a_+ - a_-}A_1, \tag{4}$$

and that for any $a\in[a_-, a_+]$, $B_0 + aB_1$ is a convex combination of A_0, A_1.

We will now define the dynamics of a v-system. The system we construct has a state vector $x_t = (z_t, y_t)$, consisting of a subvector $z_t\in\mathbf{R}^r$ and a subvector $y_t\in\mathbf{R}^{n+2}$. Let y_t^i and z_t^i stand for the ith component of y_t and z_t, respectively, and let the vector c in the definition of a v-system be such that $c^{\mathrm{T}}x_t = y_t^1$. Next, we describe the dynamics of the state vector.
Regarding z_t, we let

$$z_{t+1} = g(B_0 + v(y_t^1)B_1)z_t. \tag{5}$$

Here, g is a rational number such that

$$(m - \tfrac{1}{3})^{-1} \le g^{n+2} \le (m - \tfrac{2}{3})^{-1}. \tag{6}$$

Such a rational number exists whose size (number of bits in a binary encoding) is polynomial in m and n, and can be constructed in polynomial time. Regarding y_t, we have the following equations:

$$y_{t+1}^i = y_t^{i+1}, \quad i = 1,\dots,n+1 \tag{7}$$

and

$$y_{t+1}^{n+2} = \left(v(y_t^1) - \frac{a_- + a_+}{2}\right)\sum_{i=1}^r z_t^i. \tag{8}$$

We will show that the resulting v-system is asymptotically stable if and only if the instance of 3SAT that we started with is a "no" instance.

Suppose that we have a "no" instance of 3SAT. By the construction of Lemma 1, we have $\|A_{k_{n+2}}\cdots A_{k_2}A_{k_1}z\| \le (m-1)\|z\|$, for any vector z, and any choice of indices $k_1,\dots,k_{n+2}$. Because of Eq. (4), we see that for every value of y^1, $B_0 + v(y^1)B_1$ is a convex combination of the matrices A_0, A_1, i.e., $B_0 + v(y^1)B_1 = \gamma A_0 + (1-\gamma)A_1$, for some $\gamma\in[0,1]$. Hence, using Eq. (5),

$$\begin{aligned}\|z_{n+2}\| &\le g^{n+2}\max_{\gamma_1,\dots,\gamma_{n+2}}\|(\gamma_{n+2}A_0 + (1-\gamma_{n+2})A_1)\cdots \\ &\qquad \times(\gamma_1A_0 + (1-\gamma_1)A_1)z_0\| \\ &= g^{n+2}\max_{k_1,\dots,k_{n+2}}\|A_{k_{n+2}}\cdots A_{k_2}A_{k_1}z_0\| \\ &\le g^{n+2}(m-1)\|z_0\|.\end{aligned}$$

The first maximum is subject to the constraints $0\le\gamma_i\le 1$. It is easily shown that the maximum is attained with each γ_i equal to either zero or one, which explains the equality. Since $g^{n+2}\le(m-(2/3))^{-1}$, we conclude that $\|z_{n+2}\|\le\alpha\|z_0\|$, for some constant $\alpha<1$, from which it easily follows that z_t converges to zero. In particular, $\sum_{i=1}^r z_t^i$ converges to zero, and by inspecting Eqs. (7) and (8), we conclude that y_t also converges to zero. Since this argument was carried out for arbitrary initial conditions, we conclude that the v-system is asymptotically stable.

We now consider the case where we start with a "yes" instance of 3SAT. By the construction of Lemma 1, there exists a nonnegative nonzero integer vector $\bar z$, and some choice of indices $k_1,\dots,k_{n+2}$, such that $A_{k_{n+2}}\cdots A_{k_2}A_{k_1}\bar z = m\bar z$. Using scaling, we can assume that the components of $\bar z$ are nonnegative integer multiples of a positive integer constant K, whose value will be determined shortly. We choose the initial subvector z_0 to be any vector that satisfies

$$z_0 \ge \bar z.$$

Let M be another positive integer constant to be determined shortly. Let us say that a vector $y\in\mathbf{R}^{n+2}$ *encodes* $k_1,\dots,k_{n+2}$ if the following two conditions hold for $i = 1,\dots,n+2$:

$$y^i \ge M \quad \text{if } k_i = 1,$$

$$y^i \le -M \quad \text{if } k_i = 0.$$

We let the initial subvector y_0 be such that it encodes $k_1,\dots,k_{n+2}$. We will show that with a suitably large choice of K and M, we have $z_{n+2}\ge\bar z$ and y_{n+2} also encodes $k_1,\dots,k_{n+2}$. It will then follow (by induction) that $z_t\ge\bar z$ for all times t that are integer multiples of $n+2$, and we will have completed the proof that the v-system is not asymptotically stable.

We now set the values of the constants K and M. We first choose some $\varepsilon>0$ such that

$$\left(1 - \frac{\varepsilon}{a_+ - a_-}\right)^{n+2}\frac{m}{m-1/3} \ge 1.$$

We then choose M so that

$$v(b) \ge a^+ - \varepsilon \quad \text{if } b \ge M,$$

$$v(b) \le a^- + \varepsilon \quad \text{if } b \le -M.$$

Finally, we choose K so that

$$g^{n+2}\left(a_+ - \varepsilon - \frac{a_- + a_+}{2}\right)\left(1 - \frac{\varepsilon}{a_+ - a_-}\right)^{n+2} K \geq M.$$

For $t = 1,\ldots,n+2$, Eq. (7) yields $y^1_{t-1} = y^t_0$, which implies $v(y^1_{t-1}) = v(y^t_0)$. Since y_0 encodes $k_1,\ldots,k_{n+2}$, it follows that $v(y^1_{t-1})$ is within ε of a^+ or a_-, depending on whether k_t is 1 or 0, respectively. Suppose that $k_t = 1$. In that case, $v(y^1_{t-1}) \geq a^+ - \varepsilon$, and Eq. (4) yields

$$B_0 + v(y^1_{t-1})B_1 \geq \frac{v(y^1_{t-1}) - a_-}{a_+ - a_-} A_1 \geq \frac{a_+ - \varepsilon - a_-}{a_+ - a_-}$$

$$A_1 = \left(1 - \frac{\varepsilon}{a_+ - a_-}\right) A_{k_t}.$$

(The inequality between matrices is to be understood componentwise.) A symmetric argument also shows that if $k_t = 0$, we again have

$$B_0 + v(y^1_{t-1})B_1 \geq \left(1 - \frac{\varepsilon}{a_+ - a_-}\right) A_{k_t}.$$

This shows that we have

$$z_t \geq g\left(1 - \frac{\varepsilon}{a_+ - a_-}\right) A_{k_t} z_{t-1}, \quad t = 1,\ldots,n+2. \tag{9}$$

In particular,

$$\begin{aligned} z_{n+2} &\geq g^{n+2}\left(1 - \frac{\varepsilon}{a_+ - a_-}\right)^{n+2} A_{k_{n+2}} \cdots A_{k_1} z_0 \\ &\geq \frac{1}{m - 1/3}\left(1 - \frac{\varepsilon}{a_+ - a_-}\right)^{n+2} A_{k_{n+2}} \cdots A_{k_1} \bar{z} \\ &= \frac{1}{m - 1/3}\left(1 - \frac{\varepsilon}{a_+ - a_-}\right)^{n+2} m\bar{z} \\ &\geq \bar{z}. \end{aligned}$$

The second inequality made use of the definition of g [cf. Eq. (6)]. The equality was based on the definition of $\bar{z}$. Finally, the last inequality relied on the definition of ε.

Recall that the matrices A_0, A_1 have nonnegative integer entries. Since the entries of $\bar{z}$ are nonnegative integer multiples of K, we see that the entries of $A_{k_t} \cdots A_{k_1}\bar{z}$ have the same property, for $t = 1,\ldots,n+2$. Furthermore, for t in that range, the vector $A_{k_t} \cdots A_{k_1}\bar{z}$ must be nonzero; otherwise, we would have $m\bar{z} = A_{k_{n+2}} \cdots A_{k_1}\bar{z} = 0$, contradicting the fact that $\bar{z}$ is nonzero. Using Eq. (9), and the fact $g < 1$, we conclude that

$$\sum_{i=1}^{r} z^i_t \geq g^{n+2}\left(1 - \frac{\varepsilon}{a_+ - a_-}\right)^{n+2} K, \quad t = 1,\ldots,n+2. \tag{10}$$

Suppose that $y^1_t \geq M$. Then, $v(y^1_t) \geq a^+ - \varepsilon$. Using this inequality in Eq. (8), and using also Eq. (10), we obtain

$$y^{n+2}_{t+1} \geq g^{n+2}\left(a_+ - \varepsilon - \frac{a_- + a_+}{2}\right)\left(1 - \frac{\varepsilon}{a_+ - a_-}\right)^{n+2}$$

$$K \geq M,$$

due to the choice of K. By a symmetrical argument, if $y^1_t \leq -M$, we obtain $y^{n+2}_{t+1} \leq -M$.

We have shown that starting with $z_0 \geq \bar{z}$, and for $t = 1,\ldots,n+2$, the dynamics of y_t amount to a cyclic shift of its sign pattern, while the magnitude of each component of y_t stays above M. After $n+2$ time steps, and since y has dimension $n+2$, the same sign pattern is repeated, and y_{n+2} is again an encoding of $k_1,\ldots,k_{n+2}$. Furthermore, $z_{n+2} \geq \bar{z}$, and the same argument can be repeated. As argued earlier, this establishes that the v-system is not asymptotically stable.

We have therefore completed a reduction of the 3SAT problem to the problem of interest. The first step in the reduction, as described by Lemma 1, takes polynomial time. The remaining steps (the definition of the matrices A_0, A_1 and the constant g) also take polynomial time. Thus, the overall reduction takes polynomial time and the NP-hardness proof is complete.

Our argument has relied on the the assumption that a_+ and a_- are rational. (Without this assumption, the matrices B_0 and B_1 do not have rational entries and cannot be represented with a finite number of bits. In particular, we do not succeed in constructing an equivalent v-system in polynomial time and we do not have a legitimate reduction.) We now indicate how to generalize the proof when this assumption is relaxed. We replace a_+ and a_- in the definition of B_0 and B_1 by some rational numbers $\hat{a}_+$ and $\hat{a}_-$ that are within some $\delta > 0$ from a_+ and a_-. This is essentially the same as perturbing the matrices A_0 and A_1 to some new matrices $\hat{A}_0$ and $\hat{A}_1$ that are within $O(\delta)$ from the original matrices. Our proof has relied on the gap between the factors $m-1$ and m in Lemma 1, corresponding to the cases of "yes" and "no" instances, respectively. Under a condition of the form $\mathrm{O}((1+n\delta)^m) \leq m/(m-1)$, the gap between the two cases persists, despite the δ-perturbations of the matrices, and the reduction goes through. In addition, such a δ can be encoded with a number of bits which is polynomial in m and n, and we again have a polynomial time reduction. □

Remark. (1) Particular choices of nonconstant functions v lead to particular classes of systems for which asymptotic stability is NP-hard to decide. Consider for example the function

$$v(\alpha) = \begin{cases} +1 & \text{when } \alpha \geq 0, \\ -1 & \text{when } \alpha < 0. \end{cases}$$

This function satisfies the hypothesis of the theorem. After elementary algebraic manipulations we easily obtain:

Corollary. *The problem of deciding, for given matrices* $A_+, A_- \in \mathbf{Q}^{n\times n}$ *and vector* $c \in \mathbf{Q}^n$, *whether the system*

$$x_{t+1} = \begin{cases} A_+ x_t & \text{when } c^{\mathrm{T}} x_t \geq 0, \\ A_- x_t & \text{when } c^{\mathrm{T}} x_t < 0, \end{cases}$$

is asymptotically stable, is NP-hard.

(2) An interesting corollary of Theorem 1 is obtained by letting v be a "sigmoidal nonlinearity" of the type used in artificial neural networks. Theorem 1 implies that the stability of recurrent neural networks involving just one sigmoidal nonlinearity is NP-hard to decide.

(3) Another interesting corollary is obtained for linear systems controlled by switching controllers. A linear system $x_{t+1} = Ax_t + Bu_t$ controlled by a switching controller of the type

$$u_t = \begin{cases} K_0 x_t & \text{when } y_t \geq 0, \\ K_1 x_t & \text{when } y_t < 0, \end{cases}$$

leads to a closed-loop system

$$x_{t+1} = \begin{cases} (A + BK_0)x_t & \text{when } y_t \geq 0, \\ (A + BK_1)x_t & \text{when } y_t < 0. \end{cases}$$

From Theorem 1, we see that the stability of such systems is NP-hard to decide.

(4) A discrete-time autonomous system $f: \mathbf{R}^n \mapsto \mathbf{R}^n$ is *marginally stable* if the sequences defined by $x_{k+1} = f(x_k)$, $k = 0, 1, \ldots$, remain bounded for all initial states $x_0 \in \mathbf{R}^n$ and it is *locally stable* (asymptotically or marginally) if it is stable (asymptotically or marginally) in some neighborhood of the origin. The proof of NP-hardness of asymptotic global stability can be adapted to cover the other three cases in the four possible combinations of local/global asymptotic/marginal stability.

(5) Note that we do not know whether the asymptotic stability of v-systems is *decidable* for any or for some nonconstant function v. As mentioned earlier, this is related to the decidability of the stability of all possible sequences of products of two matrices, which is an open problem.

3. Controlled systems

A *discrete-time system* is a map $f: \mathbf{R}^n \times \mathbf{R}^m \mapsto \mathbf{R}^n: (x_t, u_t) \mapsto x_{t+1} = f(x_t, u_t)$. Let $x_\mathrm{b}, x_\mathrm{e} \in \mathbf{R}^n$ (the subscripts b and e stand for *b*eginning and *e*nd). The state x_b *can be controlled to* x_e, or, equivalently, x_e *is reachable from* x_b, if there exists some $p \geq 1$ and $u_i \in \mathbf{R}^m$ ($i = 0, \ldots, p-1$) such that the iterates

$$x_{t+1} = f(x_t, u_t), \quad t = 0, \ldots, p-1,$$

drive $x_0 = x_\mathrm{b}$ to $x_p = x_\mathrm{e}$.

A system is *controllable to* x_e if all states can be controlled to x_e, it is *reachable from* x_b if all states can be reached from x_b. In particular, the system is *null-controllable* if all states can be controlled to the origin and it is *null-reachable* if all states can be reached from the origin.

A system is *completely controllable* (or, simply, *controllable*) if all states can be controlled to all states. This notion being symmetric with respect to time, it coincides with the notion of *complete reachability*.

Asymptotic versions of these definitions are also possible by requiring the sequences to converge to the given state rather than reaching it exactly.

For linear systems the notions of complete controllability, null-reachability, and reachability from a state, are all equivalent and can be proved equivalent to the condition that the matrices A and B form a controllable pair (see, e.g., Sontag, 1990). When the matrix A is invertible, these notions furthermore coincide with those of null-controllability and of controllability to a state. Controllability of a pair of matrices can be decided in polynomial time using elementary linear algebra algorithms. For general nonlinear systems no such algorithms exist.

We define below a particular family of nonlinear systems which we consider to be the simplest possible controlled nonlinear systems, and also the simplest possible controlled hybrid systems. In Theorem 2, we analyze controllability and reachability of these systems from a computational complexity point of view.

The nth-dimensional *sign system* associated with $A_+, A_0, A_- \in \mathbf{R}^{n\times n}$ and $b, c \in \mathbf{R}^n$ is the system

$$x_{t+1} = A_{\mathrm{sgn}(c^{\mathrm{T}} x_t)} x_t + bu_t, \quad t = 0, 1, \ldots,$$

where $\mathrm{sgn}(\cdot)$ is the sign function defined in the introduction. When the control variables u_i are all zero or when $b = 0$, sign systems degenerate into autonomous systems of the form described in the previous section and for which we have shown that it is NP-hard to check asymptotic stability. It is therefore clear that asymptotic null-controllability is NP-hard to decide for sign systems. We show in Theorem 2 below that null-controllability and reachability are *undecidable* for sign systems. For proving this, we need preliminary results on Post's correspondence problem and on mortality of sets of matrices.

POST'S CORRESPONDENCE PROBLEM.

Instance: A set of pairs of words $\{(U_i, V_i): i = 1, \ldots, n\}$ over a finite alphabet.

Question: Does there exist a non-empty sequence of indices $i_1, i_2, \dots, i_k$ where $1 \le i_j \le n$, such that $U_{i_1} U_{i_2} \cdots U_{i_k} = V_{i_1} V_{i_2} \cdots V_{i_k}$?

As an illustration, consider the alphabet $\Sigma = \{1,2\}$ and the pairs of words

$U_1 = 12, \qquad V_1 = 1221,$
$U_2 = 211, \qquad V_2 = 11,$
$U_3 = 12, \qquad V_3 = 22.$

This particular instance of the correspondence problem has a solution since the words $U = U_1 U_2 U_3 U_2$ and $V = V_1 V_2 V_3 V_2$ are identical, i.e.,

$$\underbrace{12}_{U_1}\ \underbrace{211}_{U_2}\ \underbrace{12}_{U_3}\ \underbrace{211}_{U_2} = \underbrace{1221}_{V_1}\ \underbrace{11}_{V_2}\ \underbrace{22}_{V_3}\ \underbrace{11}_{V_2}.$$

On the other hand, no such correspondence is possible for the pairs

$U_1 = 12, \qquad V_1 = 1221,$
$U_2 = 21, \qquad V_2 = 121,$

since, whatever word U is on the left, the corresponding word V on the right will have a length that is strictly greater than that of U.

Post's correspondence problem is trivially decidable for one letter alphabets. Furthermore, it is easy to see that the solvability of the problem does not depend on the size of the alphabet, as long as the alphabet contains more than one letter. Post proved that the correspondence problem for an alphabet with more than one letter is undecidable (for a proof of this classical result see, e.g., Hopcroft and Ullman (1969)). In a recent contribution Matiyasevich and Sénizergues (1996) have improved this result by showing that the problem remains undecidable in the case where there are only seven pairs of words. On the other hand, the problem is known to be decidable for two pairs of words. The limit between decidability/undecidability is somewhere between three and seven pairs.

Post's correspondence problem can be used to prove a result on mortality of matrices. Let $k \ge 1$. A set $\mathscr{A}$ of square real matrices of the same dimension is *k-mortal* if there exist $A_i \in \mathscr{A}$ $(i = 1, \dots, k)$ such that $A_k \cdots A_2 A_1 = 0$. The set is *mortal* if it is k-mortal for some finite k. Paterson (1970) uses Post's correspondence problem to show that mortality of integer matrices, of size 3×3 or larger, is undecidable. This result is improved slightly in Blondel and Tsitsiklis (1997) where the following can be found:

Proposition 1. *Mortality of two integer matrices of size $n \times n$ is undecidable for $n = 6(n_p + 1)$ where n_p is any number of pairs of words for which Post's correspondence problem is undecidable.*

As mentioned earlier we can take $n_p = 7$, and thus mortality of pairs of 48×48 integer matrices is undecidable. We are now able to prove our theorem.

Theorem 2. *Let n_p be any number of pairs of words for which Post's correspondence problem is undecidable* (*we can take $n_p = 7$*).

(a) *The problem of deciding, for a given nth-dimensional sign system, whether the system is null-controllable is undecidable when $n \ge 6n_p + 7$.*

(b) *The problem of deciding, for a given nth-dimensional system and for given states $x_e, x_b \in \mathbf{Q}^n$, whether x_e can be reached from x_b, is undecidable when $n \ge 3n_p + 1$.*

Proof. (a) Let $B_0, B_1 \in \mathbf{Z}^{n \times n}$ be two arbitrary matrices of size $n \times n$. The sign system we construct has a state vector $x_t = (z_t, y_t)$ where z_t is a scalar and y_t is a vector in $\mathbf{R}^n$. Let the vector c in the definition of a sign system be such that $c^T x_t = z_t$ and let $A_- = A_0 = B_0$ and $A_+ = B_1$. We define the dynamics of the sign system by $z_{t+1} = u_t$ and $y_{t+1} = A_{\mathrm{sgn}(c^T x_t)} y_t = A_{\mathrm{sgn}(z_t)} y_t$.

For a given initial state $x_0 \in \mathbf{R}^{n+1}$ and $p \ge 1$, the state x_t is obtained by $x_t = (z_t, y_t)$ with $z_t = u_{t-1}$ and

$$y_t = A_{\mathrm{sgn}(u_{t-1})} \cdots A_{\mathrm{sgn}(u_1)} A_{\mathrm{sgn}(u_0)} A_{\mathrm{sgn}(c^T x_0)} y_0.$$

We claim that the sign system is null-controllable if and only if the matrices B_0, B_1 are mortal.

If the matrices B_0, B_1 are mortal, then the sign system is clearly null-controllable, and so this part is trivial. For the other direction, assume that the sign system is null-controllable and let e_r be the rth unit vector of $\mathbf{R}^n$. Since the system is null controllable, there exists a $k_1 \ge 0$ and a sequence $j_i \in \{-,0,+\}$, for $i = 1, \dots, k_1$ such that $A_{j_{k_1}} \cdots A_{j_2} A_{j_1} e_1 = 0$. Let $x_2 = A_{j_{k_1}} \cdots A_{j_2} A_{j_1} e_2$. By using the null-controllability assumption again, we find some $k_2 \ge 0$ and a sequence $j'_i \in \{-,0,+\}$ for $i = 1, \dots, k_2$ such that $A_{j'_{k_2}} \cdots A_{j'_2} A_{j'_1} x_2 = 0$. The product $A = A_{j'_{k_2}} \cdots A_{j'_2} A_{j'_1} A_{j_{k_1}} \cdots A_{j_2} A_{j_1}$ is such that $Ae_1 = 0$ and $Ae_2 = 0$. Continuing in the same way for all unit vectors, we eventually obtain a product A of matrices in $\{A_-, A_0, A_+\}$ such that $Ae_r = 0$ for $r = 1, \dots, n$. This implies that the set $\{A_-, A_0, A_+\}$ is mortal and thus so is the set $\{B_0, B_1\}$.

We have shown that null-controllability of the $(n+1)$th-dimensional sign system is equivalent to mortality of the set $\{B_0, B_1\}$. According to Proposition 1, the latter problem is undecidable when $n \ge 6(n_p + 1)$, hence the result.

(b) Let an instance of Post's correspondence problem be given by the pairs of words $\{(U_i, V_i)$: $i = 1, \dots, n\}$ over the alphabet $\{1,2\}$. We construct a sign system of dimension $(3n+1)$ and states x_b and x_e such that x_e can be reached from x_b if and only if the correspondence problem has a solution. Our construction is similar to the one given by Paterson (1970).

Let $|a|$ denote the length of the word a. Note that every word U_i or V_i over the alphabet $\{1,2\}$ can also be viewed as a nonnegative integer u_i or v_i, respectively. For each pair (U_i, V_i) we construct a matrix

$$W_i = \begin{pmatrix} q_i & 0 & 0 \\ 0 & s_i & 0 \\ u_i & v_i & 1 \end{pmatrix},$$

where u_i and v_i are as described above, $q_i = 10^{|U_i|}$, and $s_i = 10^{|V_i|}$. The product of the matrices W_i and W_j is given by

$$W_i W_j = \begin{pmatrix} q_i q_j & 0 & 0 \\ 0 & s_i s_j & 0 \\ u_i \oplus u_j & v_i \oplus v_j & 1 \end{pmatrix},$$

where $a \oplus b$ denotes the positive integer resulting from the concatenation of the positive integers a and b. It is therefore clear that the correspondence problem admits a solution if and only if there is a product $B_k \cdots B_1$ with $B_j \in \mathscr{W} := \{W_i\colon i = 1, \ldots, n\}$ such that

$$10^{-p} B_k \cdots B_1 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

for some $p \geq 1$ (the integer p is equal to the length of the word resulting from the correspondence). We transform this problem into a reachability problem for sign systems.

Let I_m denote the identity matrix of size m and define

$$V_1 = \operatorname{diag}(W_1, W_2, \ldots, W_n),$$

(The reason for the notation V_1 will appear shortly.)

$$S = 10^{-1} I_{3n},$$

and

$$T = \begin{pmatrix} 0 & I_{3(n-1)} \\ I_3 & 0 \end{pmatrix}.$$

All these matrices have size $3n \times 3n$. We define a sign system of dimension $(3n+1)$ by $A_+ = \operatorname{diag}(0, V_1)$, $A_0 = \operatorname{diag}(0, S)$, $A_- = \operatorname{diag}(0, T)$ and $b = c = (1 \ 0 \ \cdots \ 0)^{\mathrm{T}}$. Finally, we define the beginning and end states by

$$x_{\mathrm{b}} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} \quad \text{and} \quad x_{\mathrm{e}} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}$$

and claim that the sign system

$$x_{t+1} = A_{\operatorname{sgn}(c^{\mathrm{T}} x_t)} x_t + b u_t$$

can be driven from x_{b} to x_{e} if and only if the correspondence problem has a solution.

For notational convenience, let us partition the state vector x_t by $x_t = (z_t, y_t)$ where z_t is a scalar and y_t is a subvector of dimension $3n$. We use the corresponding decompositions of the beginning and end states $x_{\mathrm{b}} = (z_{\mathrm{b}}, y_{\mathrm{b}})$ and $x_{\mathrm{e}} = (z_{\mathrm{e}}, y_{\mathrm{e}})$. The dynamics of z_t is given by $z_0 = 1$ and $z_{t+1} = u_t$. The dynamics of y_t is given by $y_1 = V_1 y_0$ and

$$y_{t+1} = \begin{cases} V_1 y_t & \text{when } u_{t-1} > 0, \\ S y_t & \text{when } u_{t-1} = 0, \\ T y_t & \text{when } u_{t-1} < 0. \end{cases}$$

The matrix S commutes with T and V_1 and so we obtain

$$y_t = S^s V_1^{w_q} T^{t_q} \cdots V_1^{w_1} T^{t_1} V_1 y_0$$

for some $s, t_i, w_i \geq 0$. Notice that $T^n = I_{3n}$ and define

$$V_k = T^{k-1} V_1 T^{n-(k-1)}.$$

We have then

$$V_k = \operatorname{diag}(W_k, W_{k+1}, \ldots, W_n, W_1, \ldots, W_{k-1})$$

for $k = 1 \ldots, n$. Using the property $T^n = I_{3n}$ we arrive, after elementary manipulations, at

$$y_t = S^s T^{t_*} V y_0,$$

where V is a nonempty product of matrices V_i and $s, t_* \geq 0$. The matrices V_i are block-diagonal and so the blocks of V are obtained by forming non-empty products of matrices from the set $\mathscr{W}$. We can now conclude. If the Post correspondence problem has a solution, then x_{e} can be reached from x_{b} by choosing the control u_i such that $y_t = S^s V y_0$ where the last block in V is constructed from the solution of the correspondence problem and s is equal to the length of the word resulting from the correspondence. Conversely, if $y_{\mathrm{e}} = S^s T^{t_*} V y_{\mathrm{b}}$ for some nonempty product V and $s, t_* \geq 0$ then, since all $3(n-1)$ first components of y_{b} are equal to zero, and V is block-diagonal, we must have $t_* = kn$ for some $k \in \mathbf{Z}$. But then $y_{\mathrm{e}} = S^s V y_{\mathrm{b}}$ and the correspondence problem has a solution. $\square$

Remark. (1) In the proof of the first part of the theorem we use matrices and vectors that have integer entries.

Therefore null-controllability remains undecidable when matrices and vectors are constrained to have integer entries. For an integer valued sequence, convergence to zero is equivalent to equality with zero after finitely many steps. From this it follows that the asymptotic version of null-controllability is undecidable for sign systems.

(2) The class of piecewise linear systems is arguably the smallest possible class of systems that contains the classical linear systems, the finite automata, and that is closed under interconnection of such systems (see Sontag, 1996). A sign systems is a piecewise linear system with elementary partitions $c^T x > 0$, $c^T x = 0$ and $c^T x < 0$, and the results stated in Theorem 2 therefore apply to the class of piecewise linear systems.

4. Conclusion

We have shown that the stability of autonomous discrete-time systems whose dynamics are linear on each side of a hyperplane that divides the state space, is NP-hard to verify. Thus, unless P = NP, the running time of stability checking algorithms for such systems must increase faster than any polynomial in the "size" of the system.

We have also shown that null-controllability of piecewise linear systems is undecidable, even if the state space is only partitioned into three regions. This remains so even if the system has dimension 49.

The above results imply that the development of efficient algorithms for analyzing some relatively simple classes of hybrid systems appears impossible. There seem to be precious few cases of hybrid systems that are amenable to algorithmic solution, and it is certainly interesting to delineate those cases. On the other hand, with a pragmatic viewpoint, one should not hope for computational tools that always provide the correct answer and within reasonable computation time. As an alternative, we may wish to consider algorithms that can certify the stability of some hybrid systems, certify the instability of others, but can be inconclusive in some cases. Even though such algorithms do not solve the mathematical problem of deciding stability, they can certainly be a useful tool. Instead of abandoning problems for which negative complexity results are available, one may simply have to contend with partial solutions of the form just described.

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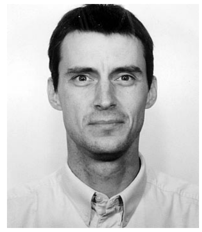

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