Partial facial reduction: simplified, equivalent SDPs via approximations of the PSD cone

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Abstract

We develop a practical semidefinite programming (SDP) facial reduction procedure that utilizes computationally efficient approximations of the positive semidefinite cone. The proposed method simplifies SDPs with no strictly feasible solution (a frequent output of parsers) by solving a sequence of easier optimization problems and could be a useful pre-processing technique for SDP solvers. We demonstrate effectiveness of the method on SDPs arising in practice, and describe our publicly-available software implementation. We also show how to find maximum rank matrices in our PSD cone approximations (which helps us find maximal simplifications), and we give a post-processing procedure for dual solution recovery that generally applies to facial-reduction-based pre-processing techniques.

1 Introduction

The feasible set of a semidefinite program (SDP) is described by the intersection of an affine subspace with the cone of matrices that are positive semidefinite (PSD). In practice, this intersection may contain no matrices that are strictly positive definite, i.e. strict feasibility may fail. This is problematic for two reasons. One, strong duality is not guaranteed. Two, the SDP (if feasible) is unnecessarily large in the sense it can be reformulated using a smaller PSD cone and a lower dimensional subspace. To see this latter point, consider the following motivating example:

Motivating example

$$\begin{array}{cl} \text{Find} & y_1, y_2, y_3 \in \mathbb{R} \\ \text{subject to} & \\ \multicolumn{2}{c}{\mathcal{A}(y) = \begin{pmatrix} y_1 & 0 & 0 \\ 0 & -y_1 & y_2 \\ 0 & y_2 & y_2 + y_3 \end{pmatrix} \succeq 0.} \end{array}$$

Taking $v = (1,1,0)^T$, it is clear that $v^T \mathcal{A}(y) v = 0$ independent of (y_1, y_2, y_3). In other words, there is no (y_1, y_2, y_3) for which $\mathcal{A}(y)$ is positive definite. It also holds that (y_1, y_2, y_3) is a feasible point of the above SDP if and only if it is a feasible point of

$$\begin{array}{cl} \text{Find} & y_1, y_2, y_3 \in \mathbb{R} \\ \text{subject to} & y_1 = y_2 = 0, y_3 \geq 0. \end{array}$$

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In other words, the above 3×3 semidefinite constraint is equivalent to linear equations and a linear inequality (i.e. a 1×1 semidefinite constraint).

While SDPs of this type may seem rare in practice, they are a frequent output of *parsers* (e.g. [25], [34]) used, for example, to formulate SDP-based relaxations of algebraic problems. In some cases, these SDPs arise because the parser does not exploit available problem structure (cf. the SDP relaxations of graph partitioning problem [45], where problem structure is carefully exploited to ensure strict feasibility). In other situations, all relevant structure is not apparent from the problem's natural high level description (which motivates the post-processing of solutions in [26]). Thus, based on their prevalence, checking for and simplifying an SDP of this type is a practically useful *pre-processing* step, assuming it can be done efficiently.

To check for (and simplify) such an SDP, one can execute the facial reduction algorithm of Borwein and Wolkowicz [10] (or a suitable variant, e.g. [31], [43]) to find a *face* of the PSD cone containing the feasible set. The desired simplifications are then obtained by rewriting the SDP as an optimization problem over this face. Unfortunately, the problem of finding a face is itself an SDP, which may be too expensive to solve in the context of pre-processing. Based on this observation, this paper presents a facial reduction algorithm modified in a simple way: rather than search over all possible faces, the algorithm searches over just a subset defined (in a particular sense) by a specified *approximation* of the PSD cone. This leads to a pre-processing technique that is easily executed: for simple approximations, the algorithm finds a face by solving just a linear or second-order cone program. This also leads to a pre-processing technique that is empirically useful: despite the use of approximations, the algorithm often significantly simplifies SDPs arising in practice, as we illustrate with examples.

This paper is organized as follows. In Section 2, we give a short derivation of a facial reduction algorithm for general conic optimization problems using basic tools from convex analysis. We then specialize this algorithm to semidefinite programs. In Section 3, we modify this algorithm to yield our technique and describe example approximations of the PSD cone. We then show how to find maximum rank solutions to conic optimization problems formulated over these approximations, which helps us find faces of minimal dimension. Section 4 shows how to reformulate a given SDP over an identified face and gives simple illustrative examples of our technique. Section 5 discusses the important issue of dual solution recovery; the results of this section are not specific to our modified facial reduction procedure and are relevant to other pre-processing techniques based on facial reduction. Section 6 describes a freely-available implementation of our procedure and Section 7 illustrates effectiveness of the method on examples arising in practice.

1.1 Prior work

Facial reduction is a common technique for reducing problem size and in this context is typically presented in an application specific manner. Techniques have been developed for SDPs arising in Euclidean distance matrix completion (Krislock and Wolkowicz [24]), protein structure identification (Alipanahi et al. [2]; Burkowski, Cheung, Wolkowicz [16]), graph partitioning (Wolkowicz and Zhao [45]), quadratic assignment (Zhao et al. [46]) and max-cut (Anjos and Wolkowicz [4]).

In addition, facial reduction can be used to ensure that strong duality holds. Indeed, this was the original motivation of the Borwein and Wolkowicz algorithm [10], which given a feasible optimization problem outputs one that is strictly feasible and hence satisfies Slater's condition. Facial reduction is also the theoretical basis for so-called extended duals, generalized dual programs for which strong duality always holds. Described by Pataki in [31], extended duals are defined for optimization problems over *nice* cones and include the Ramana dual [36] [37] for SDP.

A dual view of facial reduction was given by Luo, Sturm, and Zhang [27]. There, the authors describe a so-called conic expansion algorithm that grows the dual cone to include additional linear functionals non-negative on the feasible set. In [43], Waki and Muramatsu give a facial reduction procedure and explicitly relate it to conic expansion.

The idea of using facial reduction as a pre-processing step for SDP was described in [23] by Gruber et al. The authors note the expense of identifying lower dimensional faces as well as issues of numerical reliability that may arise. In [18], Cheung, Schurr, and Wolkowicz address the issue of numerical reliability, giving a facial reduction algorithm that identifies a *nearby* problem in a backwards stable manner. (We point out that in principle, our procedure can be made immune to numerical error for approximations that are *polyhedral* since, in this setting, it can identify faces by solving linear programs in exact arithmetic.)

Finally, our technique can be related to a philosophy put forth by Andersen and Andersen in [3] for pre-processing linear programs (LPs). There, the authors argue the best strategy for pre-processing LPs is to find *simple* simplifications *quickly*. The facial reduction algorithm we present is consistent with this philosophy in the sense that the specified approximation defines the notion of "simple" and its search complexity defines the notion of "quick."

1.2 Contributions

Partial facial reduction Our principal contribution is a technique for pre-processing semidefinite programs, based on facial reduction, that allows one to trade off problem simplifications with pre-processing effort. Given any SDP, the technique searches for an equivalent reformulation over a lower dimensional face, where a user-specified approximation of the PSD cone controls the size of this search space. Though we focus on SDP, the technique (as discussed in Section 3) applies generally to conic optimization problems—for instance, it could also be used for pre-processing second-order cone programs (SOCPs).

Maximum rank solutions Related to finding a face of minimal dimension is finding a maximum rank matrix in a subspace intersected with a specified approximation. We show (Corollary 2) how to find such a matrix when the approximation equals the Minkowski sum of faces of the PSD cone. Approximations of this type include diagonally-dominant [5], scaled diagonally-dominant, and factor-width-k [9] approximations.

Dual solution recovery We give and study a simple algorithm for dual solution recovery (Algorithm 3), a critical post-processing step for linear programming that, to our knowledge, has not been explored for conic optimization. The presented procedure applies generally to conic optimization problems pre-processed using facial reduction techniques; in other words, it is not specific to SDP and does not depend on the approximations we introduce. Since pre-processing may remove duality gaps, dual solution recovery is *not always possible*. Hence, we give conditions (Conditions 1 and 2) characterizing success of the procedure for SDPs—the class of conic optimization problem of primary interest.

Software implementation We have implemented our technique in MATLAB in a set of scripts we call `frlib`, available at `www.mit.edu/~fperment`. If interfaced directly, the code takes as input SDPs in SeDuMi format [39]. It can also be interfaced via the parser YALMIP [25].

2 Background on facial reduction

In this section, we define our notation, collect basic facts and definitions, and describe faces of the PSD cone. We then derive a facial reduction procedure for general conic optimization problems and specialize it to semidefinite programs.

2.1 Notation and preliminaries

Let $\mathcal{E}$ denote a finite-dimensional vector space over $\mathbb{R}$ with inner product $\langle \cdot, \cdot \rangle$. For a subset $\mathcal{S}$ of $\mathcal{E}$, let $\operatorname{lin} \mathcal{S} \subseteq \mathcal{E}$ denote the linear span of elements in $\mathcal{S}$ and let $\mathcal{S}^\perp \subseteq \mathcal{E}$ denote the orthogonal complement of $\operatorname{lin} \mathcal{S}$. For $y \in \mathcal{E}$, let $\operatorname{lin} y$ denote $\operatorname{lin}\{y\}$ and let $y^\perp$ denote $\{y\}^\perp$. A convex cone $\mathcal{K}$ is a convex subset of $\mathcal{E}$ (not necessarily full dimensional) that satisfies

$$x \in \mathcal{K} \Rightarrow \alpha x \in \mathcal{K} \quad \forall \alpha \geq 0.$$

The dual cone of $\mathcal{K}$, denoted $\mathcal{K}^*$, is the set of linear functionals non-negative on $\mathcal{K}$:

$$\mathcal{K}^* := \{y : \langle y, x \rangle \geq 0 \quad \forall x \in \mathcal{K}\}.$$

A face $\mathcal{F}$ of a convex cone $\mathcal{K}$ is a convex subset that satisfies

$$\frac{a+b}{2} \in \mathcal{F} \text{ and } a, b \in \mathcal{K} \Rightarrow a, b \in \mathcal{F}.$$

A face is *proper* if it is non-empty and not equal to $\mathcal{K}$. Faces of convex cones are also convex cones, and the relation "is a face of" is transitive; if $\mathcal{F}_2$ is a face of $\mathcal{F}_1$ and $\mathcal{F}_3$ is a face of $\mathcal{F}_2$, then $\mathcal{F}_3$ is face of $\mathcal{F}_1$. For any $s \in \mathcal{K}^*$, the set $\mathcal{K} \cap s^\perp$ is a face of $\mathcal{K}$. Further, if $\mathcal{K}$ is closed, it holds that $(\mathcal{K} \cap s^\perp)^* = \overline{\mathcal{K}^* + \operatorname{lin} s}$ (where we let $\overline{\mathcal{S}}$ denote the closure of a set $\mathcal{S}$).

For discussions specific to semidefinite programming, we let $\mathbb{S}^n$ denote the vector space of $n \times n$ symmetric matrices and $\mathbb{S}^n_+ \subseteq \mathbb{S}^n$ denote the convex cone of matrices that are positive semidefinite. We will use capital letters to denote elements of $\mathbb{S}^n$ to emphasize that they are matrices. For $A, B \in \mathbb{S}^n$, we let $A \cdot B$ denote the trace inner product $\operatorname{Tr} AB$. Finally, we let $A \succeq 0$ (resp. $A \succ 0$) denote the condition that A is positive semidefinite (resp. positive definite).

2.2 Faces of $\mathbb{S}^n_+$

A set is a face of $\mathbb{S}^n_+$ if and only if it equals the set of all $n \times n$ PSD matrices with range contained in a given d-dimensional subspace [5] [30]. Using this fact, one can describe a proper face $\mathcal{F}$ (and the dual cone $\mathcal{F}^*$) using an invertible matrix $(U, V) \in \mathbb{R}^{n \times n}$, where the range of $U \in \mathbb{R}^{n \times d}$ equals this subspace and the range of $V \in \mathbb{R}^{n \times n-d}$ equals $(\operatorname{range} U)^\perp$. We collect such descriptions in the following.

Lemma 1. *A non-zero, proper face of $\mathbb{S}^n_+$ is a set $\mathcal{F}$ of the form*

$$\begin{aligned} \mathcal{F} &:= \left\{ (U, V) \begin{pmatrix} W & 0 \\ 0 & 0 \end{pmatrix} (U, V)^T : W \in \mathbb{S}^d_+ \right\} && (1) \\ &= \left\{ X \in \mathbb{S}^n : U^T X U \succeq 0, U^T X V = 0, V^T X V = 0 \right\}, && (2) \end{aligned}$$

where $(U, V) \in \mathbb{R}^{n \times n}$ is an invertible matrix satisfying $U^T V = 0$ (i.e. $\operatorname{range} V = (\operatorname{range} U)^\perp$). Moreover, the dual cone $\mathcal{F}^*$ satisfies

$$\begin{aligned} \mathcal{F}^* &= \left\{ (U, V) \begin{pmatrix} W & Z \\ Z^T & R \end{pmatrix} (U, V)^T : W \in \mathbb{S}^d_+, Z \in \mathbb{R}^{d \times n-d}, R \in \mathbb{S}^{n-d} \right\} && (3) \\ &= \left\{ X \in \mathbb{S}^n : U^T X U \in \mathbb{S}^d_+ \right\}. && (4) \end{aligned}$$

Here, (1) and (3) represent a face $\mathcal{F}$ and its dual cone $\mathcal{F}^*$ in terms of *generators* whereas (2) and (4) represent these sets in terms of *constraints*. Either representation can be preferred depending on context. Based off of (1), we will often refer to a face $\mathcal{F}$ using the notation $U\mathbb{S}^d_+U^T$.

2.3 Facial reduction of conic optimization problems

Conic optimization problems The feasible set of a conic optimization problem is described by the intersection of an affine subspace $\mathcal{A}$ with a convex cone $\mathcal{K}$, where both $\mathcal{A}$ and $\mathcal{K}$ are subsets of a vector space $\mathcal{E}$. If one defines the affine subspace $\mathcal{A}$ in terms of a linear map $A : \mathbb{R}^m \to \mathcal{E}$ and a point $c \in \mathcal{E}$, i.e.

$$\mathcal{A} := \{c - Ay : y \in \mathbb{R}^m\},$$

one can express a conic optimization problem as follows:

$$\text{maximize } \langle b, y\rangle \text{ subject to } c - Ay \in \mathcal{K},$$

where $b \in \mathbb{R}^m$ defines a linear objective function. This conic optimization problem is *feasible* if $\mathcal{A} \cap \mathcal{K}$ is non-empty and *strictly feasible* if $\mathcal{A} \cap \operatorname{relint} \mathcal{K}$ is non-empty.

Reformulation over a face If a conic optimization problem is feasible but not strictly feasible, it can be reformulated as an optimization problem over a lower dimensional face of $\mathcal{K}$. This fact will follow from the following lemma:

Lemma 2. *Let $\mathcal{K}$ be a convex cone and $\mathcal{A}$ be an affine subspace for which $\mathcal{A}\cap\mathcal{K}$ is non-empty. The following statements are equivalent.*

1. *$\mathcal{A} \cap \operatorname{relint} \mathcal{K}$ is empty.*
2. *There exists $s \in \mathcal{K}^* \setminus \mathcal{K}^\perp$ for which the hyperplane $s^\perp$ contains $\mathcal{A}$.*

Proof. To see (1) implies (2), note the main separation theorem (Theorem 11.3) of Rockafellar [38] states a hyperplane exists that *properly* separates the sets $\mathcal{A}$ and $\mathcal{K}$ if the intersection of their relative interiors is empty. Using Theorem 11.7 of Rockafellar, we can additionally assume this hyperplane passes through the origin since $\mathcal{K}$ is a cone. In other words, if $\mathcal{A} \cap \operatorname{relint} \mathcal{K}$ is empty, there exists s satisfying

$$\begin{array}{lll} \langle s, x\rangle \leq 0 & \text{for all} & x \in \mathcal{A} \\ \langle s, x\rangle \geq 0 & \text{for all} & x \in \mathcal{K} \\ \langle s, x\rangle \neq 0 & \text{for some} & x \in \mathcal{A} \cup \mathcal{K}. \end{array}$$

We will show that $\langle s, x\rangle = 0$ for all $x \in \mathcal{A}$, which will establish statement (2). Let x_0 denote a point in $\mathcal{A} \cap \mathcal{K}$ and let $\mathcal{T}$ be a subspace for which $\mathcal{A} = x_0 + \mathcal{T}$. Clearly, $\langle s, x_0\rangle = 0$. Since $\langle s, x_0\rangle$ vanishes, we must have that $\langle s, x\rangle \leq 0$ for all $x \in \mathcal{T}$. But $\mathcal{T}$ is a subspace, therefore $\langle s, -x\rangle \leq 0$ also must hold. Thus, $\langle s, x\rangle = 0$ for all $x \in \mathcal{T}$ and $s^\perp$ contains $\mathcal{A}$. Since $\langle s, x\rangle$ vanishes for all $x \in \mathcal{A}$, $\langle s, x\rangle \neq 0$ holds for some $x \in \mathcal{K}$. This establishes that s is not in $\mathcal{K}^\perp$ and completes the proof.

To see that (2) implies (1), suppose $s \in \mathcal{K}^* \setminus \mathcal{K}^\perp$ exists and suppose for contradiction there exists an $x_0 \in \mathcal{A} \cap \operatorname{relint} \mathcal{K}$. Since $\mathcal{K}^\perp$ is the orthogonal complement of $\operatorname{lin} \mathcal{K}$, we can decompose s as $s = s_1 + s_2$, where $s_1 \in \operatorname{lin} \mathcal{K}$ and s_2 is in $\mathcal{K}^\perp$. Note that s_1 is also in $\mathcal{K}^*$, s_1 is non-zero, and $\langle s_1, x_0\rangle = \langle s, x_0\rangle = 0$. Since the affine hull of $\mathcal{K}$ equals the subspace $\operatorname{lin} \mathcal{K}$, we must have that $x_0 - \epsilon s_1$ is in $\mathcal{K}$ for some $\epsilon > 0$. This implies

$$\langle s_1, x_0 - \epsilon s_1\rangle = -\epsilon||s_1||^2 \geq 0,$$

which cannot hold for any $\epsilon > 0$. Hence, no $x \in \mathcal{A} \cap \operatorname{relint} \mathcal{K}$ exists.

$\square$

The vector s given by statement (2) is called a *reducing certificate* for $\mathcal{A} \cap \mathcal{K}$. Notice intersection with $s^\perp$ leaves $\mathcal{A} \cap \mathcal{K}$ unchanged. Letting $\mathcal{F}$ denote the face $\mathcal{K} \cap s^\perp$ therefore yields the following equivalent optimization problem:

$$\text{maximize } \langle b, y \rangle \text{ subject to } c - Ay \in \mathcal{F}.$$

Since faces of convex cones are also convex cones, this simplification can be repeated if one can find a reducing certificate for $\mathcal{A} \cap \mathcal{F}$. Indeed, by taking $\mathcal{F}_{i+1} = \mathcal{F}_i \cap s_i^\perp$ for an $s_i \in \mathcal{F}_i^* \setminus \mathcal{F}_i^\perp$ orthogonal to $\mathcal{A}$, one can find a chain of faces $\mathcal{F}_i$

$$\mathcal{K} = \mathcal{F}_0 \supset \mathcal{F}_1 \supset \cdots \supset \mathcal{F}_{n-1} \supset \mathcal{F}_n$$

that contain $\mathcal{A} \cap \mathcal{K}$. Producing this sequence is called *facial reduction*. An explicit algorithm for producing this sequence is given in [31], which we essentially reproduce in Algorithm 1.

Algorithm 1: Facial reduction algorithm. Computes a sequences of faces $\mathcal{F}_i$ of the cone $\mathcal{K}$ containing $\mathcal{A} \cap \mathcal{K}$, where $\mathcal{A}$ is an affine subspace.

begin
  Initialize: $\mathcal{F}_0 \leftarrow \mathcal{K}, i = 0$
  repeat
    1. *Find reducing certificate,* i.e. solve the feasibility problem

$$\begin{array}{rl} \text{Find} & s_i \in \mathcal{F}_i^* \setminus \mathcal{F}_i^\perp \\ \text{subject to} & s_i^\perp \text{ contains } \mathcal{A} \end{array} \qquad (\star)$$

    2. *Compute new face,* i.e. set $\mathcal{F}_{i+1} = \mathcal{F}_i \cap s_i^\perp$

    3. *Increment counter i*

  until $(\star)$ is infeasible;
end

Finding reducing certificates To execute the facial reduction algorithm (Algorithm 1), one must solve a feasibility problem $(\star)$ at each iteration to find $s_i \in \mathcal{F}_i^* \setminus \mathcal{F}_i^\perp$ orthogonal to $\mathcal{A}$. It turns out this feasibility problem is also a conic optimization problem. To see this, recall the definition of $\mathcal{A}$ from above (i.e. $\mathcal{A} := \{c - Ay : y \in \mathbb{R}^m\}$), let $A^* : \mathcal{E} \to \mathbb{R}^m$ denote the adjoint of A and pick x_0 in the relative interior of $\mathcal{F}_i$. The solutions to $(\star)$ are (up to scaling) the solutions to:

$$\begin{array}{rll} \text{Find } s_i & & \\ \text{subject to} & s_i \in \mathcal{F}_i^*, \langle s_i, x_0 \rangle = 1 & (\text{i.e. } s_i \in \mathcal{F}_i^* \setminus \mathcal{F}_i^\perp) \\ & A^* s_i = 0, \langle c, s_i \rangle = 0 & (\text{i.e. } s_i^\perp \text{ contains } \mathcal{A}\ ). \end{array} \qquad (5)$$

That $s_i^\perp$ contains $\mathcal{A}$ if and only if the second line of constraints holds can be shown using the standard identity $(\operatorname{range} A)^\perp = \operatorname{null} A^*$. Correctness of the first line of constraints arises from the following corollary of Lemma 2:

Corollary 1. *Let $\mathcal{K}$ be a convex cone, let s be an element of $\mathcal{K}^*$, and let x be any element of* $\operatorname{relint}\mathcal{K}$. *Then, $s \in \mathcal{K}^* \setminus \mathcal{K}^\perp$ if and only if $\langle s, x\rangle > 0$.*

Proof. The if direction is obvious. To see the other direction, suppose s is in $\mathcal{K}^* \setminus \mathcal{K}^\perp$ and $\langle s, x\rangle = 0$. Applying Lemma 2, this implies $\{x\} \cap \operatorname{relint}\mathcal{K}$ is empty, a contradiction. ☐

Discussion We make a few concluding remarks about the algorithm. First, it terminates after finitely many steps, since the dimension of $\mathcal{F}_i$ drops at each iteration. Second, if the algorithm terminates after n iterations, then $\mathcal{A} \cap \operatorname{relint}\mathcal{F}_n$ is non-empty, a simple consequence of Lemma 2. In other words, a reformulation of the original problem over the face $\mathcal{F}_n$ is strictly feasible.

Remark 1. *Throughout this section, we have assumed the given problem is feasible. If the facial reduction algorithm (as presented) is applied to a problem that is infeasible, it will identify faces $\mathcal{F}_i$ for which the sets $\mathcal{A} \cap \mathcal{F}_i$ are also empty, leading to an equivalent problem that is also infeasible. Though it is possible to modify the algorithm to detect infeasibility (see, e.g., [43]), we forgo this to simplify presentation.*

2.4 Facial reduction of semidefinite programs

In this section, we develop a version of the facial reduction algorithm (Algorithm 1) for semidefinite programs, i.e. we consider the case where the cone $\mathcal{K} = \mathbb{S}^n_+$ and the vector space $\mathcal{E} = \mathbb{S}^n$. This procedure, given explicitly by Algorithm 2, represents each face $\mathcal{F}_i$ as a set of the form $U_i \mathbb{S}^{d_i}_+ U_i^T$ (with $d_i \leq n$) for an appropriate rectangular matrix U_i (leveraging the description of faces given by Lemma 1). It finds reducing certificates $S_i \in \mathbb{S}^n$ by solving a semidefinite program over $\mathbb{S}^{d_i}_+$ and it computes a new face $\mathcal{F}_{i+1} := \mathcal{F}_i \cap S_i^\perp$ by finding a basis for the null space of particular matrix (related to the reducing certificate). It can be applied to an SDP of the following form:

$$\begin{array}{ll} \text{maximize} & y^T b \\ \text{subject to} & C - \sum_{j=1}^m y_j A_j \in \mathbb{S}^n_+, \end{array}$$

where C and A_j are fixed symmetric matrices defining an affine subspace $\mathcal{A}$ of $\mathbb{S}^n$:

$$\mathcal{A} := \left\{ C - \sum_{j=1}^m y_j A_j : y \in \mathbb{R}^m \right\}.$$

We now explain the basic steps of Algorithm 2 in more detail.

Step one: find reducing certificate At each iteration i, Algorithm 2 finds a reducing certificate $S_i \in \mathcal{F}_i^* \setminus \mathcal{F}_i^\perp$ for $\mathcal{A} \cap \mathcal{F}_i$, where $\mathcal{F}_i$ denotes the face $U_i \mathbb{S}^{d_i}_+ U_i^T$. This is done by solving conic optimization problem (5) specialized to the case $\mathcal{K} = \mathbb{S}^n_+$. This specialization appears as SDP ($\star$), where we've used (4) of Lemma 1 to describe $\mathcal{F}_i^*$ and the point $U_i U_i^T \in \operatorname{relint}\mathcal{F}_i$ to describe $\mathcal{F}_i^* \setminus \mathcal{F}_i^\perp$.

Step two: compute new face The second step of Algorithm 2 computes a new face by intersecting $\mathcal{F}_i$ with the subspace $S_i^\perp$. Computing this intersection can be done using a matrix $B \in \mathbb{R}^{d \times r}$ with range equal to $\operatorname{null} U_i^T S_i U_i$. Explicitly, we have that $\mathcal{F}_i \cap S_i^\perp = U_i B \mathbb{S}^r_+ B^T U_i^T$, as shown in the next lemma.

Algorithm 2: Facial reduction algorithm for an SDP. Computes a sequences of faces $\mathcal{F}_i := U_i \mathbb{S}_+^{d_i} U_i^T$ of $\mathbb{S}_+^n$ containing $\mathcal{A} \cap \mathbb{S}_+^n$, where $\mathcal{A} := \left\{C - \sum_{j=1}^m y_j A_j : y \in \mathbb{R}^m\right\}$.

Initialize: $U_0 = I_{n\times n}, d_0 = n,\ i = 0$

repeat

1. *Find reducing certificate* S_i, i.e. solve the SDP

$$\begin{array}{lll} \text{Find} & S_i \in \mathbb{S}^n & \\ \text{subject to} & U_i^T S_i U_i \in \mathbb{S}_+^d, U_i U_i^T \cdot S_i = 1 & (i.e.\ S_i \in \mathcal{F}_i^* \setminus \mathcal{F}_i^\perp) \quad (\star) \\ & C \cdot S_i = 0,\ A_j \cdot S_i = 0\ \ \forall j \in \{1, \ldots, m\} & (i.e.\ S_i^\perp \text{ contains } \mathcal{A}) \end{array}$$

2. *Compute new face,* i.e. find basis B for $\operatorname{null} U_i^T S_i U_i$, set U_{i+1} equal to $U_i B$, and set d_{i+1} equal to $\dim \operatorname{null} U_i^T S_i U_i$.

3. *Increment counter* i

until $(\star)$ is infeasible;

Lemma 3. *For $U \in \mathbb{R}^{n\times d}$, let $\mathcal{F}$ denote the set $U\mathbb{S}_+^d U^T$ and let $S \in \mathbb{S}^n$ and $B \in \mathbb{R}^{d\times r}$ satisfy*

$$U^T S U \succeq 0, \qquad \operatorname{range} B = \operatorname{null} U^T S U.$$

The following relationship holds:

$$\mathcal{F} \cap S^\perp = U B \mathbb{S}_+^r B^T U^T.$$

Proof. The containment $\supseteq$ is obvious. To see the other containment, let UXU^T be an element of $\mathcal{F} \cap S^\perp$ for some $X \succeq 0$. Taking inner product with S yields

$$UXU^T \cdot S = X \cdot U^T S U = 0.$$

Since $X \succeq 0$ and $U^T S U \succeq 0$, the inner product $X \cdot U^T S U$ vanishes if and only if $\operatorname{range} X$ is contained in $\operatorname{null} U^T S U$ (see, for example, Proposition 2.7.1 of [30]). In other words, X is in the face $B\mathbb{S}_+^r B^T$ of $\mathbb{S}_+^d$, completing the proof. □

Discussion We now make a few comments about Algorithm 2. Variants of this algorithm arise by using different descriptions of $\mathcal{F}_i$ or by using different descriptions of the affine subspace $\mathcal{A}$. If, for instance, one represents $\mathcal{A}$ as the set of X solving the equations $A_j \cdot X = b_j$ for $j \in \{1, \ldots, m\}$, then the set of S_i orthogonal to $\mathcal{A}$ equals the set

$$\left\{ \sum_{j=1}^m y_j A_j : y \in \mathbb{R}^m, y^T b = 0 \right\}. \tag{6}$$

Hence, to apply Algorithm 2 to SDPs defined by equations $A_j \cdot X = b_j$, one simply replaces the constraints $C \cdot S_i = 0, A_j \cdot S_i = 0$ with membership in (6). We also note from Lemma 1 that one can represent $\mathcal{F}_i$ and $\mathcal{F}_i^*$ using a sequence of invertible matrices (U_i, V_i), which could be a more convenient description depending on implementation or the representation of $\mathcal{A}$. Finally, we note a barrier to executing Algorithm 2, and the general procedure given in Algorithm 1, is the cost of finding reducing certificates. In the next section, we describe a modification of these algorithms that allows one to reduce this cost through use of approximations.

3 Approach

3.1 Partial facial reduction

Each iteration of the general facial reduction algorithm (Algorithm 1) finds a reducing certificate by solving the feasibility problem ($\star$). Though the reducing certificate identifies a lower dimensional face, this benefit must be traded off with the cost of solving ($\star$). In this section, we propose a method for managing this trade-off. Specifically, we describe a method for reducing the complexity of the feasibility problem ($\star$) at the cost of only *partially* simplifying the given conic optimization problem.

Our method is as follows. Letting $\mathcal{F}_i$ denote the current face at iteration i of Algorithm 1, we approximate $\mathcal{F}_i$ with a user-specified convex cone $\mathcal{F}_{i,outer}$ that satisfies:

1. $\mathcal{F}_i \subseteq \mathcal{F}_{i,outer}$ (which implies $\mathcal{F}_{i,outer}^* \subseteq \mathcal{F}_i^*$)
2. $\operatorname{lin} \mathcal{F}_i = \operatorname{lin} \mathcal{F}_{i,outer}$ (i.e. $\mathcal{F}_{i,outer}^\perp = \mathcal{F}_i^\perp$)
3. $\mathcal{F}_{i,outer}^*$ has low search complexity,

where the first two conditions ensure that $\mathcal{F}_{i,outer}^* \setminus \mathcal{F}_{i,outer}^\perp$ is a subset of $\mathcal{F}_i^* \setminus \mathcal{F}_i^\perp$. Using the approximation $\mathcal{F}_{i,outer}$, we then modify the feasibility problem ($\star$) to search over this subset:

$$\begin{array}{ll} \text{Find} & s_i \in \mathcal{F}_{i,outer}^* \setminus \mathcal{F}_{i,outer}^\perp \subseteq \mathcal{F}_i^* \setminus \mathcal{F}_i^\perp \\ \text{subject to} & s_i^\perp \text{ contains } \mathcal{A}. \end{array} \qquad (\star)$$

By construction, a solution $s_i \in \mathcal{F}_{i,outer}^* \setminus \mathcal{F}_{i,outer}^\perp$ to the modified feasibility problem is a solution to the original; hence, $\mathcal{F}_i \cap s_i^\perp$ is a face of $\mathcal{F}_i$ (and the cone $\mathcal{K}$) containing $\mathcal{A} \cap \mathcal{K}$. Further, the approximation $\mathcal{F}_{i,outer}$ can be chosen such that the search complexity of $\mathcal{F}_{i,outer}^*$ matches desired pre-processing effort. In other words, the algorithm correctly identifies a face at cost specified by the user.

3.1.1 Existence of reducing certificates

Because we have introduced the approximation $\mathcal{F}_{i,outer}$, the algorithm may not find a reducing certificate (and hence a lower dimensional face) even if $\mathcal{A} \cap \operatorname{relint} \mathcal{F}_i$ is empty; hence, the algorithm may not find a face of minimal dimension. This leads to the following question: when will the modified feasibility problem ($\star$) have a solution? Since we have chosen $\mathcal{F}_{i,outer}$ to be a convex cone, we can use Lemma 2 to answer this question. Under the assumption that $\mathcal{A} \cap \mathcal{F}_{i,outer}$ is non-empty, this lemma states feasibility of ($\star$) is now *equivalent* to emptiness of $\mathcal{A} \cap \operatorname{relint} \mathcal{F}_{i,outer}$. In other words, the modified feasibility problem ($\star$) has a solution if and only if a *relaxation* of the problem of interest is not strictly feasible. Figure 1 illustrates a situation when this condition holds and when it fails for two different subspaces.

3.1.2 Approximating faces of $\mathbb{S}_+^n$

To apply this idea to SDP, and to suitably modify the SDP facial reduction algorithm (Algorithm 2), we need a way of approximating faces of $\mathbb{S}_+^n$. To see how this can be done, let $\mathcal{F}$ denote a face

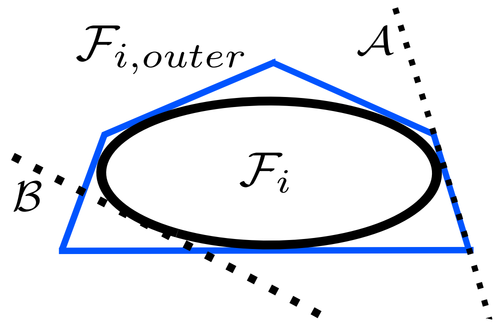

Figure 1: Illustrates when the facial reduction algorithm successfully finds a face when modified to use the approximation $\mathcal{F}_{i,outer}$. If the feasible set is contained in $\mathcal{A} \cap \mathcal{F}_i$, the procedure succeeds: $\mathcal{A} \cap \operatorname{relint} \mathcal{F}_{i,outer}$ is empty. In contrast, if the feasible set is contained in $\mathcal{B} \cap \mathcal{F}_i$, the procedure fails: $\mathcal{B} \cap \operatorname{relint} \mathcal{F}_{i,outer}$ is non-empty.

$U\mathbb{S}^d_+U^T$ of $\mathbb{S}^n_+$ for some $U \in \mathbb{R}^{n\times d}$. An approximation $\mathcal{F}_{outer}$ can be defined using an approximation $\hat{\mathbb{S}}^d_+$ of $\mathbb{S}^d_+$. Moreover, the search complexity of $\mathcal{F}^*_{outer}$ depends on the search complexity of $\hat{\mathbb{S}}^d_+$. Consider the following (whose proof is straight-forward and omitted):

Lemma 4. *Let $\hat{\mathbb{S}}^d_+ \subseteq \mathbb{S}^d$ be a convex cone containing $\mathbb{S}^d_+$. For $U \in \mathbb{R}^{n\times d}$, let $\mathcal{F}_{outer}$ and $\mathcal{F}$ denote the sets $U\hat{\mathbb{S}}^d_+U^T$ and $U\mathbb{S}^d_+U^T$, respectively. The following statements are true.*

1. $\mathcal{F} \subseteq \mathcal{F}_{outer}$.
2. $\operatorname{lin} \mathcal{F} = \operatorname{lin} \mathcal{F}_{outer}$
3. $\mathcal{F}^*_{outer} = \left\{ X \in \mathbb{S}^n : U^T X U \in (\hat{\mathbb{S}}^d_+)^* \right\}$.

Based on this lemma, we conclude to modify Algorithm 2, it suffices to replace the PSD constraint of SDP ($\star$) with membership in $(\hat{\mathbb{S}}^d_+)^*$, where $\hat{\mathbb{S}}^d_+$ is a cone outer-approximating $\mathbb{S}^d_+$. Example approximations are explored in the next section.

3.2 Approximations of $\mathbb{S}^d_+$

In this section, we explore an outer approximation $\mathcal{C}(\mathbb{W})$ of $\mathbb{S}^d_+$ parametrized by a set $\mathbb{W}$ of $d \times k$ rectangular matrices. The parametrization is chosen such that the dual cone $\mathcal{C}(\mathbb{W})^*$ equals the *Minkowski sum* of faces $W_i\mathbb{S}^k_+W_i^T$ of $\mathbb{S}^d_+$ for $W_i \in \mathbb{W}$. It is defined below:

Lemma 5. *For a set $\mathbb{W} := \left\{W_1, W_2, \dots, W_{|\mathbb{W}|}\right\}$ of $d \times k$ matrices, let $\mathcal{C}(\mathbb{W})$ denote the following convex cone:*

$$\mathcal{C}(\mathbb{W}) := \left\{ X \in \mathbb{S}^d : W_i^T X W_i \in \mathbb{S}^k_+ \quad i = 1, \dots, |\mathbb{W}| \right\}.$$

The dual cone $\mathcal{C}(\mathbb{W})^*$ satisfies

$$\mathcal{C}(\mathbb{W})^* = \left\{ \sum_{i=1}^{|\mathbb{W}|} W_i X_i W_i^T : X_i \in \mathbb{S}^k_+ \right\}, \tag{7}$$

and the following inclusions hold:

$$\mathcal{C}(\mathbb{W})^* \subseteq \mathbb{S}^d_+ \subseteq \mathcal{C}(\mathbb{W}).$$

$\mathcal{C}(\mathbb{W})$	$\mathcal{C}(\mathbb{W})^*$	Search	$\lvert\mathbb{W}\rvert$
$X_{ii} \geq 0$	Non-negative diagonal ($\mathcal{D}^d$)	LP	$\mathcal{O}(d)$
$X_{ii} \geq 0$, $X_{jj} + X_{ii} \pm 2X_{ij} \geq 0$	Diagonally-dominant ($\mathcal{DD}^d$)	LP	$\mathcal{O}(d^2)$
2×2 principal sub-matrices psd	Scaled diagonally-dominant ($\mathcal{SDD}^d$)	SOCP	$\mathcal{O}(d^2)$
$k \times k$ principal sub-matrices psd	Factor width-k ($\mathcal{FW}_k^d$)	SDP	$\mathcal{O}(\binom{d}{k})$

Table 1: Example outer and inner approximations of $\mathbb{S}_+^d$, the search algorithm for $\mathcal{C}(\mathbb{W})^*$, and the cardinality of the set $\mathbb{W}$.

Proof. The inclusions are obvious from the definitions of $\mathcal{C}(\mathbb{W})^*$ and $\mathcal{C}(\mathbb{W})$ (as is the fact that $\mathcal{C}(\mathbb{W})$ is a convex cone). It remains to show correctness of (7). To show this, let $\mathcal{T}$ denote the set on the right-hand side of (7). It is easy to check that $\mathcal{T}^* = \mathcal{C}(\mathbb{W})$, which implies $\mathcal{T}^{**} = \mathcal{C}(\mathbb{W})^*$. Since $\mathcal{T}$ is a convex cone (as is easily checked), $\mathcal{T}^{**}$ equals the closure of $\mathcal{T}$. The result therefore follows by showing $\mathcal{T}$ is closed. To see this, note that $\mathcal{T}$ equals the Minkowski sum of closed cones $W_i \mathbb{S}_+^k W_i^T$. For matrices $Z_i \in W_i \mathbb{S}_+^k W_i^T$, we have that $\sum_{i=1}^{|\mathbb{W}|} Z_i = 0$ only if $Z_i = 0$ for each i. This shows that $\sum_{i=1}^{|\mathbb{W}|} Z_i = 0$ only if Z_i is in the lineality space of $W_i \mathbb{S}_+^k W_i^T$. Direct application of the closedness criteria Corollary 9.1.3 of Rockafellar [38] shows $\mathcal{T}$ is closed. $\square$

Since the modification to the SDP facial reduction algorithm (Algorithm 2) will involve searching over $\mathcal{C}(\mathbb{W})^*$ (as indicated by Lemma 4), we will investigate $\mathcal{C}(\mathbb{W})$ by studying the dual cone $\mathcal{C}(\mathbb{W})^*$. We first make a few comments regarding the search complexity of $\mathcal{C}(\mathbb{W})^*$ for different choices of $\mathbb{W}$. Note when $k = 1$, each W_j in $\mathbb{W}$ is a vector and $\mathcal{C}(\mathbb{W})^*$ is the conic hull of a finite set of rank one matrices. In other words, $\mathcal{C}(\mathbb{W})^*$ is *polyhedral* and can be described by *linear programming.* When $k = 2$, the set $\mathcal{C}(\mathbb{W})^*$ is defined by 2×2 semidefinite constraints and can hence be described by *second-order cone programming* (SOCP). This follows since each $X_i \in \mathbb{S}_+^2$ can be expressed using scalars a, b, c constrained as follows:

$$X_i = \begin{pmatrix} a+b & c \\ c & a-b \end{pmatrix} \succeq 0 \quad \Leftrightarrow \quad a \geq 0 \quad \text{and } a^2 \geq b^2 + c^2. \tag{8}$$

Example choices for $\mathcal{C}(\mathbb{W})^*$ are now given. As we will see, well-studied approximations of $\mathbb{S}_+^d$ can be expressed as sets of the form $\mathcal{C}(\mathbb{W})^*$.

3.2.1 Examples

Example choices for $\mathcal{C}(\mathbb{W})^*$ are given in Table 1 along with the cardinality of the set $\mathbb{W}$ that yields each entry. Included are $d \times d$ non-negative diagonal matrices $\mathcal{D}^d$, diagonally-dominant matrices $\mathcal{DD}^d$, scaled diagonally-dominant matrices $\mathcal{SDD}^d$ as well as matrices $\mathcal{FW}_k^d$ with *factor-width* [9] bounded by k. These sets satisfy

$$\mathcal{D}^d = \mathcal{FW}_1^d \subseteq \mathcal{DD}^d \subseteq \mathcal{SDD}^d = \mathcal{FW}_2^d \subseteq \mathcal{FW}_3^d \subseteq \cdots \subseteq \mathcal{FW}_d^d = \mathbb{S}_+^d,$$

and the sets $\mathcal{D}^d$ and $\mathcal{DD}^d$ are polyhedral. Details on each entry follow.

3.2.2 Non-negative diagonal matrices ($\mathcal{D}^d$)

A simple choice for $\mathcal{C}(\mathbb{W})^* \subseteq \mathbb{S}_+^d$ is the set of non-negative diagonal matrices:

$$\mathcal{D}^d := \left\{ X \in \mathbb{S}^d : X_{ii} \geq 0, \;\; X_{ij} = 0 \;\; \forall i \neq j \right\}.$$

The set $\mathcal{D}^d$ contains non-negative combinations of matrices $w_i w_i^T$, where w_i is a permutation of $(1,0,\ldots,0,0)^T$. In other words, the set $\mathcal{D}^d$ corresponds to the set $\mathcal{C}(\mathbb{W})^*$ if we take

$$\mathbb{W} = \left\{(1,0,\ldots,0,0)^T, (0,1,\ldots,0,0)^T, \ldots, (0,0,\ldots,0,1)^T\right\}.$$

3.2.3 Diagonally-dominant matrices ($\mathcal{DD}^d$)

Another well studied choice for $\mathcal{C}(\mathbb{W})^*$ is cone of symmetric diagonally-dominant matrices with non-negative diagonal entries [5]:

$$\mathcal{DD}^d := \left\{ X \in \mathbb{S}^d : X_{ii} \geq \sum_{j \neq i} |X_{ij}| \right\}.$$

This set is polyhedral. The *extreme rays* of $\mathcal{DD}^d$ are matrices of the form $w_i w_i^T$, where w_i is any permutation of

$$(1,0,0,\ldots,0)^T, (1,1,0,\ldots,0)^T, \text{ or } (1,-1,0,\ldots,0)^T.$$

Taking $\mathbb{W}$ equal to the set of all such permutations gives $\mathcal{C}(\mathbb{W})^* = \mathcal{DD}^d$. This representation makes the inclusion $\mathcal{DD}^d \subseteq \mathbb{S}_+^d$ obvious. We also see that $\mathcal{DD}^d$ contains $\mathcal{D}^d$.

3.2.4 Scaled diagonally-dominant matrices ($\mathcal{SDD}^d$)

A non-polyhedral generalization of $\mathcal{DD}^d$ is the set of *scaled diagonally-dominant matrices* $\mathcal{SDD}^d$. This set equals all matrices obtained by pre- and post-multiplying diagonally-dominant matrices by diagonal matrices with strictly positive diagonal entries:

$$\mathcal{SDD}^d := \left\{ DTD : D \in \mathcal{D}^d, D_{ii} > 0, T \in \mathcal{DD}^d \right\}.$$

The set $\mathcal{SDD}^d$ can be equivalently defined as the set of matrices that equal the sum of PSD matrices non-zero only on a 2×2 principal sub-matrix (Theorem 9 of [9]). As an explicit example, we have that $\mathcal{SDD}^3$ are all matrices X of the form

$$X = \underbrace{\begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{11} & a_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{X_1} + \underbrace{\begin{pmatrix} b_{11} & 0 & b_{13} \\ 0 & 0 & 0 \\ b_{13} & 0 & b_{33} \end{pmatrix}}_{X_2} + \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & c_{22} & c_{23} \\ 0 & c_{23} & c_{33} \end{pmatrix}}_{X_3},$$

where a_{ij}, b_{ij}, and c_{ij} are scalars chosen such that X_1, X_2 and X_3 are PSD. In general, $\mathcal{SDD}^d$ equals $\mathcal{C}(\mathbb{W})^*$ when $\mathbb{W}$ equals the set of $d \times 2$ matrices W for which $W^T X W$ returns a 2×2 principal sub-matrix of X. For $\mathcal{SDD}^3$, we have

$$\mathcal{SDD}^3 = C(\{W_1, W_2, W_3\})^* = \left\{ \sum_{i=1}^{3} W_i X_i W_i^T : X_i \in \mathbb{S}_+^2 \right\},$$

where

$$W_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \quad W_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \quad W_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Also note from (8) that $\mathcal{SDD}^d$ can be represented using second-order cone constraints. This latter fact is used in recent work of Ahmadi and Majumdar [1] to define an SOCP-based method for testing polynomial non-negativity. (A similar LP-based method is also presented in [1] that incorporates $\mathcal{DD}^d$.)

3.2.5 Factor-width-k matrices

A generalization of $\mathcal{SDD}^d$ (and diagonal matrices $\mathcal{D}^d$) arises from notion of *factor-width* [9]. The factor-width of a matrix X is the smallest integer k for which X can be written as the sum of PSD matrices that are non-zero only on a single $k \times k$ principal sub-matrix.

Letting $\mathcal{FW}_k^d$ denote the set of $d \times d$ matrices of factor-width no greater than k, we have that $\mathcal{SDD}^d = FW_2^d$ and $\mathcal{D}^d = \mathcal{FW}_1^d$. To represent $\mathcal{FW}_k^d$ as a cone of the form $\mathcal{C}(\mathbb{W})^*$, we set $\mathbb{W}$ to be the set of $d \times k$ matrices W_j for which $W_j^T X W_j$ returns a $k \times k$ principal sub-matrix of X. Note that there are $\binom{d}{k}$ such matrices, so a complete parametrization of $\mathcal{FW}_k^d$ is not always practical using this representation. Also note $\mathcal{FW}_k^d$ equals $\mathbb{S}_+^d$ when $k = d$.

3.2.6 Corresponding outer approximations

We briefly discuss the outer approximation $\mathcal{C}(\mathbb{W})$ corresponding to the discussed examples for $\mathcal{C}(\mathbb{W})^*$. To summarize, if $\mathcal{C}(\mathbb{W})^*$ is the cone of non-negative diagonal matrices $\mathcal{D}^d$, then $\mathcal{C}(\mathbb{W})$ is the cone of matrices whose diagonal entries are non-negative. If $\mathcal{C}(\mathbb{W})^*$ is the cone of non-negative diagonally-dominant matrices $\mathcal{DD}^d$, then $\mathcal{C}(\mathbb{W})$ is the set of matrices X for which $w_i^T X w_i \geq 0$, where $w_i w_i^T$ is an extreme ray of $\mathcal{DD}^d$ (given in Section 3.2.3). If $\mathcal{C}(\mathbb{W})^*$ is the set of scaled diagonally-dominant matrices $\mathcal{SDD}^d$, then $\mathcal{C}(\mathbb{W})$ is the set of matrices with positive semidefinite 2×2 principal sub-matrices. Finally, if $\mathcal{C}(\mathbb{W})^*$ equals $\mathcal{FW}_k^d$, the set of matrices with factor-width bounded by k, then $\mathcal{C}(\mathbb{W})$ is the set of matrices with positive semidefinite $k \times k$ principal sub-matrices. We see as $\mathcal{C}(\mathbb{W})^*$ grows larger, the constraints defining $\mathcal{C}(\mathbb{W})$ become more restrictive, equaling a positive semidefinite constraint when $d = k$.

3.3 Finding faces of minimal dimension/rank maximizing reducing certificates

Suppose $\mathcal{F}_i := U_i \mathbb{S}_+^{d_i} U_i^T$ is the current face at iteration i of the SDP facial reduction algorithm (Algorithm 2). Further suppose $\mathcal{F}_{i,outer} := U_i \mathcal{C}(\mathbb{W}_i) U_i^T$ approximates $\mathcal{F}_i$ per the discussion in Section 3.1 (for some specified set of rectangular matrices $\mathbb{W}_i$). The following question is natural: how can one find a reducing certificate S_i that minimizes the dimension of the face $\mathcal{F}_{i+1} := \mathcal{F}_i \cap S_i^\perp$ when S_i is constrained to $\mathcal{F}_{i,outer}^* \setminus \mathcal{F}_{i,outer}^\perp$? Using Lemma 3, it is easy to see this problem is solved by finding a solution to

$$
\begin{array}{lll}
\text{Find} & S_i \in \mathbb{S}^n & \\
\text{subject to} & U_i^T S_i U_i \in \mathcal{C}(\mathbb{W}_i)^* & (i.e.\ S_i \in \mathcal{F}_{i,outer}^*) \\
 & C \cdot S_i = 0,\ A_j \cdot S_i = 0 \ \ \forall j \in \{1, \ldots, m\} & (i.e.\ S_i^\perp \text{ contains } \mathcal{A})
\end{array}
$$

that maximizes the rank of $U_i^T S_i U_i$. In this section, we give a method for finding solutions of this type.

To ease notation, we drop the subscript i and also consider a more general question: how does one find maximum rank matrices in the set $\mathcal{M} \cap \mathcal{C}(\mathbb{W})^*$, when $\mathcal{M}$ is an arbitrary linear subspace? (In the above, $\mathcal{M}$ is the subspace $\left\{U_i^T S_i U_i : C \cdot S_i = 0, A_j \cdot S_i = 0\right\}$.) An answer to this question arises from the next two lemmas.

Lemma 6. *Let $\mathcal{M}$ be a subspace of $\mathbb{S}^d$. If $X^\star := \sum_{i=1}^{|\mathbb{W}|} W_i X_i^\star W_i^T$ maximizes $\sum_{i=1}^{|\mathbb{W}|} \operatorname{rank} X_i$ over $\mathcal{M} \cap \mathcal{C}(\mathbb{W})^*$, then $X^\star$ maximizes $\operatorname{rank} X$ over $\mathcal{M} \cap \mathcal{C}(\mathbb{W})^*$.*

Proof. We will argue the kernel of $X^\star$ is contained in the kernel of any $X \in \mathcal{M} \cap \mathcal{C}(\mathbb{W})^*$, which immediately implies $\operatorname{rank} X^\star \geq \operatorname{rank} X$.

To begin, we first argue for any $X = \sum_{i=1}^{|\mathbb{W}|} W_i X_i W_i^T \in \mathcal{M} \cap \mathcal{C}(\mathbb{W})^*$ that $\operatorname{null} X_i^\star \subseteq \operatorname{null} X_i$ for all $i \in \{1, \ldots, |\mathbb{W}|\}$. To see this, first note that for any $X \in \mathcal{M} \cap \mathcal{C}(\mathbb{W})^*$ the matrix

$$X^\star + X = \sum_{i=1}^{|\mathbb{W}|} W_i (X_i^\star + X_i) W_i^T$$

is also in $\mathcal{M} \cap \mathcal{C}(\mathbb{W})^*$ and satisfies $\operatorname{rank} X_i^\star + X_i \geq \operatorname{rank} X_i^\star$. Now suppose for some $d \in \{1, \ldots, |\mathbb{W}|\}$ that $\operatorname{null} X_d^\star \not\subseteq \operatorname{null} X_d$. This implies that $\operatorname{null} X_d^\star + X_d = \operatorname{null} X_d^\star \cap \operatorname{null} X_d \subset \operatorname{null} X_d^\star$ which in turn implies $\operatorname{rank}(X_d^\star + X_d) > \operatorname{rank} X_d^\star$. But this contradicts our assumption that $X^\star$ maximizes $\sum_i \operatorname{rank} X_i$. Hence, $\operatorname{null} X_i^\star \subseteq \operatorname{null} X_i$ for all $i \in \{1, \ldots, |\mathbb{W}|\}$.

Now suppose an $X \in \mathcal{M} \cap \mathcal{C}(\mathbb{W})^*$ exists for which $X^\star w = 0$ but $Xw \neq 0$ for some w. Since $Xw = 0$ if and only if $X_i W_i^T w = 0$ for all i, we must have for some d that $W_d^T w$ is in the kernel of $X_d^\star$ but not in the kernel of X_d. But we have already established that $\operatorname{null} X_d^\star \subseteq \operatorname{null} X_d$. Hence, w cannot exist. We therefore have that $\operatorname{null} X^\star \subseteq \operatorname{null} X$ for any $X \in \mathcal{M} \cap \mathcal{C}(\mathbb{W})^*$, which completes the proof. $\square$

We can use this condition to formulate an SDP whose optimal solutions yield maximum rank matrices of $\mathcal{M} \cap \mathcal{C}(\mathbb{W})^*$. To maximize $\sum_{i=1}^{|\mathbb{W}|} \operatorname{rank} X_i$, we introduce matrices T_i constrained such that their traces $\operatorname{Tr} T_i$ lower bound $\operatorname{rank} X_i$. We then optimize the sum of their traces.

Lemma 7. *Let $\mathcal{M}$ be a subspace of $\mathbb{S}^d$. A matrix X maximizing $\sum_{i=1}^{|\mathbb{W}|} \operatorname{rank} X_i$ over $\mathcal{M} \cap \mathcal{C}(\mathbb{W})^*$ is given by any optimal solution (X, X_i, T_i) to the following SDP:*

$$\begin{array}{lrl} \text{maximize} \sum_{i=1}^{|\mathbb{W}|} \operatorname{Tr} T_i & & \\ \text{subject to} & X \in \mathcal{M}, & \\ & X = \sum_{i=1}^{|\mathbb{W}|} W_i X_i W_i^T & \text{i.e. } X \in \mathcal{C}(\mathbb{W})^* \\ & X_i \succeq T_i & \forall i \in \{1, \ldots, |\mathbb{W}|\} \\ & I \succeq T_i \succeq 0 & \forall i \in \{1, \ldots, |\mathbb{W}|\}. \end{array} \tag{9}$$

Proof. Let $r_{\max}$ equal the maximum of $\sum_{i=1}^{|\mathbb{W}|} \operatorname{rank} X_i$ over the set of feasible X_i. We will show at optimality $\sum_{i=1}^{|\mathbb{W}|} \operatorname{rank} X_i = r_{\max}$.

To begin, the constraint $I \succeq T_i \succeq 0$ implies the eigenvalues of T_i are less than one. Hence, $\operatorname{rank} T_i \geq \operatorname{Tr} T_i$. Since $X_i \succeq T_i$, we also have $\operatorname{rank} X_i \geq \operatorname{rank} T_i$. Thus, any feasible (X_i, T_i) pair satisfies

$$r_{\max} \geq \sum_{i=1}^{|\mathbb{W}|} \operatorname{rank} X_i \geq \sum_{i=1}^{|\mathbb{W}|} \operatorname{rank} T_i \geq \sum_{i=1}^{|\mathbb{W}|} \operatorname{Tr} T_i. \tag{10}$$

Now note for any feasible (X, X_i) we can pick $\alpha > 0$ and construct a feasible point $(\alpha X, \alpha X_i, \hat{T}_i)$ that satisfies $\sum_{i=1}^{|\mathbb{W}|} \operatorname{Tr} \hat{T}_i = \sum_{i=1}^{|\mathbb{W}|} \operatorname{rank} X_i$; if X_i has eigen-decomposition $\sum_j \lambda_j u_j u_j^T$ for $\lambda_j > 0$, simply take $\hat{T}_i = \sum_j u_j u_j^T$ and α equal to

$$\max \bigcup_i \left\{ \frac{1}{\lambda} : \lambda \text{ is a positive eigenvalue of } X_i \right\}.$$

Hence, some feasible point $(\hat{X}, \hat{X}_i, \hat{T}_i)$ satisfies $\sum_{i=1}^{|\mathbb{W}|} \operatorname{Tr} \hat{T}_i = r_{\max}$. Therefore, the optimal (X, X_i, T_i) satisfies

$$\sum_{i=1}^{|\mathbb{W}|} \operatorname{Tr} T_i \geq r_{\max}.$$

Combining this inequality with (10) yields that at optimality

$$\sum_{i=1}^{|\mathbb{W}|} \operatorname{Tr} T_i = \sum_{i=1}^{|\mathbb{W}|} \operatorname{rank} X_i = r_{\max},$$

which completes the proof. $\square$

Combining the previous two lemmas shows how to maximize rank over $\mathcal{M} \cap \mathcal{C}(\mathbb{W})^*$:

Corollary 2. *A matrix $X \in \mathcal{M} \cap \mathcal{C}(\mathbb{W})^*$ of maximum rank is given by any optimal solution (X, X_i, T_i) to the SDP* (9).

Maximum rank solutions for polyhedral approximations We next illustrate how the search for maximum rank solutions simplifies when $\mathcal{C}(\mathbb{W})^*$ is polyhedral. Recall if $\mathbb{W}$ is a set of vectors, i.e. if $k = 1$, then $\mathcal{C}(\mathbb{W})^*$ is the conic hull of a finite set of rank one matrices. In other words, $\mathcal{C}(\mathbb{W})^*$ is the set of matrices of the form $\sum_{i=1}^{|\mathbb{W}|} \lambda_i w_i w_i^T$ for $\lambda_i \geq 0$ and $w_i \in \mathbb{W}$. In this case, SDP (9) simplifies into the following linear program.

Corollary 3. *A matrix $X \in \mathcal{M} \cap \mathcal{C}(\mathbb{W})^*$ of maximum rank is given by any optimal solution (X, λ, t) to the following LP:*

$$\begin{array}{lrl} \text{maximize} & \sum_{i=1}^{|\mathbb{W}|} t_i & \\ \text{subject to} & X \in \mathcal{M} & \\ & X = \sum_{i=1}^{|\mathbb{W}|} \lambda_i w_i w_i^T, & \text{i.e. } X \in \mathcal{C}(\mathbb{W})^* \\ & \lambda_i \geq t_i & \forall i \in \{1, \ldots, |\mathbb{W}|\} \\ & 1 \geq t_i \geq 0 & \forall i \in \{1, \ldots, |\mathbb{W}|\}. \end{array} \tag{11}$$

An alternative approach We briefly mention an alternative to SDP (9) for maximizing $\sum_{i=1}^{|\mathbb{W}|} \operatorname{rank} X_i$. Notice that membership in $\mathcal{C}(\mathbb{W})^*$ can be expressed using a semidefinite constraint on a block-diagonal matrix, where maximizing the rank of this matrix is equivalent to maximizing $\sum_{i=1}^{|\mathbb{W}|} \operatorname{rank} X_i$. We conclude $\sum_{i=1}^{|\mathbb{W}|} \operatorname{rank} X_i$ is maximized by finding a maximum rank solution to a particular block-diagonal SDP, which can be done using interior point methods (since solutions in the relative interior of the feasible set are solutions of maximum rank). Note, however, that this alternative approach does not permit use of the simplex method when $\mathcal{C}(\mathbb{W})^*$ is polyhedral, since the simplex method produces solutions on the boundary of the feasible set. In contrast, the simplex method can be used to solve LP (11), the specialization of SDP (9) to polyhedral $\mathcal{C}(\mathbb{W})^*$.

3.4 Explicit modifications of the SDP facial reduction algorithm (Algorithm 2)

The results of this section are now combined to modify Algorithm 2. Specifically, we introduce an approximation $\mathcal{F}_{i,outer} := U_i \mathcal{C}(\mathbb{W}_i) U_i^T$ of the face $\mathcal{F}_i := U_i \mathbb{S}_+^{d_i} U_i^T$ at each iteration i, where $\mathbb{W}_i := \left\{W_1, \ldots W_{|\mathbb{W}_i|}\right\}$ is some specified set of rectangular matrices. A reducing certificate $S_i \in \mathcal{F}_{i,outer}^*$ is then found that maximizes the rank of $U_i^T S_i U_i \in \mathcal{C}(\mathbb{W}_i)^*$ by replacing SDP $(\star)$ of Algorithm 2 with the following:

$$
\begin{array}{ll}
\text{maximize} \sum_{k=1}^{|\mathbb{W}_i|} \operatorname{Tr} T_k & \\
\text{subject to} & \\
& \begin{aligned}
U_i^T S_i U_i = \textstyle\sum_{k=1}^{|\mathbb{W}_i|} W_k \bar{S}_k W_k^T, &\quad \text{i.e. } U_i^T S_i U_i \in \mathcal{C}(\mathbb{W}_i)^* \\
U_i U_i^T \cdot S_i > 0 & \\
\bar{S}_k \succeq T_k &\quad \forall k \in \{1, \ldots, |\mathbb{W}_i|\} \\
I \succeq T_k \succeq 0 &\quad \forall k \in \{1, \ldots, |\mathbb{W}_i|\} \\
C \cdot S_i = 0,\ A_j \cdot S_i = 0 &\quad \forall j \in \{1, \ldots, m\}.
\end{aligned}
\end{array} \qquad (\star)
$$

Here, the decision variables are S_i and $T_k, \bar{S}_k$ (for $k \in \{1, \ldots, |\mathbb{W}_i|\}$) and the first two constraints are equivalent to the condition that $S_i \in \mathcal{F}_{i,outer}^* \setminus \mathcal{F}_{i,outer}^\perp$ (since $U_i U_i^T \in \operatorname{relint} \mathcal{F}_{i,outer}$). To maximize the rank of $U_i^T S_i U_i$, we have applied Corollary 2, taking $\mathcal{M}$ equal to the subspace $\{U_i^T S U_i : C \cdot S = 0, A_j \cdot S = 0\}$. Note the strict inequality $U_i U_i^T \cdot S_i = \operatorname{Tr} U_i^T S_i U_i > 0$ is satisfied by any non-zero matrix in $\mathcal{M} \cap \mathcal{C}(\mathbb{W}_i)^*$; hence, in practice one can remove this inequality and instead verify that $U_i^T S_i U_i \neq 0$ holds at optimality, i.e. one can verify $\mathcal{M} \cap \mathcal{C}(\mathbb{W}_i)^*$ contains a non-zero matrix.

Again note the complexity of solving this modified problem is controlled by $\mathbb{W}_i$. When $\mathbb{W}_i$ contains $d_i \times 1$ rectangular matrices (e.g. $\mathcal{C}(\mathbb{W}_i)^*$ equals $\mathcal{D}^{d_i}$ or $\mathcal{DD}^{d_i}$), the modified problem is a linear program. When $\mathbb{W}_i$ contains $d_i \times 2$ rectangular matrices (e.g. $\mathcal{C}(\mathbb{W}_i)^*$ equals $\mathcal{SDD}^{d_i}$), it is an SOCP.

4 Formulation of reduced problems and illustrative examples

The facial reduction algorithm for SDP (Algorithm 2) identifies a face of $\mathbb{S}_+^n$ that can be used to formulate an equivalent SDP. In this section, we show how to formulate this SDP and then give simple examples illustrating the basic steps of Algorithm 2. In these examples, we also modify Algorithm 2 to use approximations in the manner described in Section 3.

4.1 Formulation of reduced problems

Algorithm 2 identifies a face $\mathcal{F} := U\mathbb{S}_+^d U^T$ (where $U \in \mathbb{R}^{n \times d}$ is a fixed matrix with linearly independent columns and $d \leq n$) containing the intersection of $\mathbb{S}_+^n$ with an affine subspace $\mathcal{A} := \{C - \sum_{i=1}^m y_i A_i : y \in \mathbb{R}^m\}$. Letting $V \in \mathbb{R}^{n \times n-d}$ denote a matrix whose columns form a basis for $\operatorname{null} U^T$, we can reformulate the original SDP (reproduced below):

$$
\begin{array}{ll}
\text{maximize} & y^T b \\
\text{subject to} & C - \sum_{i=1}^m y_i A_i \in \mathbb{S}_+^n
\end{array}
$$

explicitly over $\mathcal{F}$ as follows:

$$
\begin{array}{ll}
\text{maximize} & y^T b \\
\text{subject to} & U^T(C - \sum_{i=1}^m y_i A_i)U \in \mathbb{S}_+^d \\
& U^T(C - \sum_{i=1}^m y_i A_i)V = 0 \\
& V^T(C - \sum_{i=1}^m y_i A_i)V = 0,
\end{array}
$$

where we have used a representation of $\mathcal{F}$ given by Lemma 1. Here, we see the reduced program is a semidefinite program over $\mathbb{S}_+^d$ described by linear equations and $d \times d$ matrices $U^T C U$ and $U^T A_i U$.

4.2 Illustrative Examples

4.2.1 Example with diagonal approximations ($\mathcal{D}^d$)

In this example, we modify Algorithm 2 to use diagonal approximations; i.e. at iteration i, the face $\mathcal{F}_i := U_i \mathbb{S}_+^{d_i} U_i^T$ is approximated by the set $\mathcal{F}_{i,outer} = U_i \mathcal{C}(\mathbb{W}_i) U_i^T$, where $\mathcal{C}(\mathbb{W}_i)^*$ equals $\mathcal{D}^{d_i}$, the set of $d_i \times d_i$ matrices that are non-negative and diagonal. A reducing certificate S_i is found in $\mathcal{F}_{i,outer}^*$, the set of matrices X for which $U_i^T X U_i$ is in $\mathcal{D}^{d_i}$. We apply the algorithm to the following SDP:

$$
\begin{array}{l}
\text{Find } y \in \mathbb{R}^4 \\
\text{subject to}
\end{array}
$$

$$
\mathcal{A}(y) = \begin{pmatrix} y_1 & 0 & 0 & 0 & 0 \\ 0 & -y_1 & y_2 & 0 & 0 \\ 0 & y_2 & y_2 - y_3 & 0 & 0 \\ 0 & 0 & 0 & y_3 & 0 \\ 0 & 0 & 0 & 0 & y_4 \end{pmatrix} \in \mathbb{S}_+^5.
$$

Taking U_0 equal to the identity matrix and the initial face equal to $\mathcal{F}_0 = U_0 \mathbb{S}_+^5 U_0$, we seek a matrix S_0 orthogonal to $\mathcal{A}(y)$ (for all y) for which $U_0^T S_0 U_0$ is non-negative and diagonal. An S_0 satisfying this constraint and a basis B for $\operatorname{null} U_0^T S_0 U_0$ is given by:

$$
S_0 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
$$

Taking $U_1 = U_0 B = B$, yields the face $\mathcal{F}_1 = U_1 \mathbb{S}_+^3 U_1^T$, i.e. the set of PSD matrices in $\mathbb{S}_+^5$ with vanishing first and second rows/cols.

Continuing to the next iteration, we seek a matrix S_1 orthogonal to $\mathcal{A}(y)$ for which $U_1^T S_1 U_1$ is non-negative and diagonal. An S_1 satisfying this constraint and a basis B for $\operatorname{null} U_1^T S_1 U_1$ is given by:

$$
S_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.
$$

Setting $U_2 = U_1 B$ gives the face $\mathcal{F}_2 = U_2 \mathbb{S}_+^1 U_2^T$, where $U_2 = (0, 0, 0, 0, 1)^T$.

Terminating the algorithm, we now formulate a reduced SDP over $\mathcal{F}_2$. Letting V denote a basis for $\operatorname{null} U_2^T$ yields:

$$
\begin{array}{ll}
\text{Find } y \in \mathbb{R}^4 & \\
\text{subject to} & U_2^T \mathcal{A}(y) U_2 \in \mathbb{S}_+^1 \\
& U_2^T \mathcal{A}(y) V = 0 \\
& V^T \mathcal{A}(y) V = 0,
\end{array}
$$

which simplifies to

$$
\begin{array}{ll}
\text{Find } y \in \mathbb{R}^4 & \\
\text{subject to} & y_4 \geq 0 \\
& y_1 = y_2 = y_3 = 0.
\end{array}
$$

Existence of reducing certificates As promised by Lemma 2 and the discussion in Section 3.1.1, existence of a reducing certificate in $\mathcal{F}^*_{i,outer}$ implies emptiness of $\mathcal{A}(y) \cap \operatorname{relint} \mathcal{F}_{i,outer}$. Indeed, $\mathcal{A}(y)$ is contained in $\operatorname{relint} \mathcal{F}_{0,outer}$ only if the inequalities

$$y_1 \geq 0 \qquad -y_1 \geq 0$$

are strictly satisfied, which clearly cannot hold. Similarly, $\mathcal{A}(y)$ is contained in $\operatorname{relint} \mathcal{F}_{1,outer}$ only if $y_1 = y_2 = 0$ and the inequalities

$$y_3 \geq 0 \qquad y_2 - y_3 \geq 0$$

are strictly satisfied, which again cannot hold.

4.2.2 Example with diagonally-dominant approximations ($\mathcal{DD}^d$)

In this next example, we modify Algorithm 2 to use diagonally-dominant approximations; i.e. at iteration i, the face $\mathcal{F}_i := U_i \mathbb{S}^{d_i}_+ U_i^T$ is approximated by the set $\mathcal{F}_{i,outer} = U_i \mathcal{C}(\mathbb{W}_i) U_i^T$, where $\mathcal{C}(\mathbb{W}_i)^*$ equals $\mathcal{DD}^{d_i}$, the set of $d_i \times d_i$ matrices that are diagonally-dominant. A reducing certificate S_i is found in $\mathcal{F}^*_{i,outer}$, the set of matrices X for which $U_i^T X U_i$ is in $\mathcal{DD}^{d_i}$. We apply the algorithm to the SDP

$$\begin{array}{l} \text{Find } y \in \mathbb{R}^3 \\ \text{subject to} \end{array}$$

$$\mathcal{A}(y) = \begin{pmatrix} 1 & -y_1 & 0 & -y_3 \\ -y_1 & 2y_2 - 1 & y_3 & 0 \\ 0 & y_3 & 2y_1 - 1 & -y_2 \\ -y_3 & 0 & -y_2 & 1 \end{pmatrix} \in \mathbb{S}^4_+,$$

and execute just a single iteration of facial reduction. Taking U_0 equal to the identity, a matrix S_0 orthogonal to $\mathcal{A}$ for which $U_0^T S_0 U_0$ is diagonally-dominant and a basis B for $\operatorname{null} U_0^T S_0 U_0$ is given by

$$S_0 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \quad B = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{pmatrix}.$$

Taking $U_1 = U_0 B = B$, yields the face $\mathcal{F}_1 = U_1 \mathbb{S}^2_+ U_1^T$. Terminating the algorithm and constructing the reduced SDP using a matrix V satisfying $\operatorname{range} V = \operatorname{null} U_1^T$ imposes the linear constraints that $y_1 = y_2 = 1, y_3 = 0$; i.e. the reduced SDP has a feasible set consisting of a single point.

Existence of reducing certificates As was the case in the previous example, existence of a reducing certificate in $\mathcal{F}^*_{i,outer}$ implies emptiness of $\mathcal{A}(y) \cap \operatorname{relint} \mathcal{F}_{i,outer}$. At the first (and only) iteration, membership of $\mathcal{A}(y)$ in $\mathcal{F}_{0,outer}$ implies $w_k^T \mathcal{A}(y) w_k \geq 0$, where $w_k w_k^T$ is any extreme ray of $\mathcal{DD}^4$. Taking $w_1 = (1,1,0,0)^T$ and $w_2 = (0,0,1,1)^T$, we have that $\mathcal{A}(y)$ is contained in $\operatorname{relint} \mathcal{F}_{0,outer}$ only if the inequalities

$$\begin{aligned} w_1^T \mathcal{A}(y) w_1 &= 2y_2 - 2y_1 \geq 0 \\ w_2^T \mathcal{A}(y) w_2 &= 2y_1 - 2y_2 \geq 0 \end{aligned}$$

are strictly satisfied, which cannot hold.

5 Recovery of dual solutions

In this section we address a question that is relevant to any pre-processing technique based on facial reduction, i.e. it does not depend in any way on the approximations introduced in Section 3. Specifically, how (and when) can one recover solutions to the original dual problem? To elaborate, consider the following primal-dual pair[1] for a general conic optimization problem over a closed, convex cone $\mathcal{K}$:

$$
\begin{array}{lllll}
(P): & & \quad & (D): & \\
\quad \text{maximize} & \langle b, y\rangle & & \quad \text{minimize} & \langle c, x\rangle \\
\quad \text{subject to} & c - Ay \in \mathcal{K} & & \quad \text{subject to} & A^*x = b \\
 & & & & x \in \mathcal{K}^*,
\end{array}
$$

and suppose the general facial reduction algorithm (Algorithm 1) is applied to the primal problem (P). The reduced primal-dual pair is written over the identified face $\mathcal{F}$ and its dual cone $\mathcal{F}^*$ as follows:

$$
\begin{array}{lllll}
(R/P): & & \quad & (R/D): & \\
\quad \text{maximize} & \langle b, y\rangle & & \quad \text{minimize} & \langle c, x\rangle \\
\quad \text{subject to} & c - Ay \in \mathcal{F} & & \quad \text{subject to} & A^*x = b \\
 & & & & x \in \mathcal{F}^*.
\end{array}
$$

Since (by construction) $\mathcal{F}$ contains $c - Ay$ for any feasible point y of (P), it is clear that any solution to (R/P) solves (P). Conversely, it is clear a solution x to the dual program (R/D) is not necessarily even a feasible point of (D) since $\mathcal{K}^* \subseteq \mathcal{F}^*$. While recovering a solution to (D) from a solution to (R/D) may seem in general hopeless, the facial reduction algorithm produces reducing certificates $s_i \in \mathcal{F}_i^*$, where

$$\mathcal{K}^* = \mathcal{F}_0^* \subset \mathcal{F}_1^* \subset \cdots \subset \mathcal{F}_N^* = \mathcal{F}^*,$$

that can be leveraged to make recovery possible. This leads to the following problem statement:

Problem 1 (Recovery of dual solutions)**.** *Given a solution x to (R/D), reducing certificates $s_0, \ldots, s_{N-1}$, i.e. given s_i for which*

$$
\begin{array}{rcll}
\langle c, s_i\rangle & = & 0 & \\
A^* s_i & = & 0 & \\
s_i & \in & \mathcal{F}_i^* \setminus \mathcal{F}_i^{\perp} & \\
\mathcal{F}_{i+1} & := & \mathcal{F}_i \cap s_i^{\perp} & \qquad (\textit{which implies} \quad \mathcal{F}_{i+1}^* = \overline{\mathcal{F}_i^* + \operatorname{lin} s_i}) \\
\mathcal{F}_0 & := & \mathcal{K}, \;\; \mathcal{F} := \mathcal{F}_N, &
\end{array}
$$

find a solution to (D).

To devise an algorithm for solving this problem, we observe that each s_i is a feasible direction for (R/D) that does not increase the dual objective $\langle c, x\rangle$. We also observe that $\mathcal{F}_{i+1}^* = \overline{\mathcal{F}_i^* + \operatorname{lin} s_i}$ (since $\mathcal{K}$, and hence $\mathcal{F}_i$, is closed). This implies if $\mathcal{F}_i^* + \operatorname{lin} s_i$ is closed, one could, for any $x \in \mathcal{F}_{i+1}^*$, find an α such that $x + \alpha s_i$ is in $\mathcal{F}_i^*$. We conclude if $\mathcal{F}_i^* + \operatorname{lin} s_i$ were closed for each i, then a

[1] This designation of primal (P) and dual (D), while standard in facial reduction literature, is opposite the convention used by semidefinite solvers such as SeDuMi. We will switch to the convention favored by solvers when we discuss our software implementation in Section 6.

solution to (D) could be constructed using a sequence of line searches. In other words, the following algorithm would successfully recover a solution to (D):

Algorithm 3: Recovery of dual solutions

Input: A solution $x \in \mathcal{F}^*$ to the reduced dual (R/D) and reducing certificates $s_0, \ldots, s_{N-1}$
Output: A solution x to the original dual (D) or flag indicating failure.
for $i \leftarrow N-1$ down to 0 **do**

1. Using a line search, find α s.t. $x + \alpha s_i \in \mathcal{F}_i^*$.
2. If no α exists, return FAIL. Else, set $x \leftarrow x + \alpha s_i$.

end

The following properties of this algorithm can be stated immediately:

Lemma 8. *Algorithm 3 has the following properties:*

1. Sufficient condition for recovery. *Algorithm 3 succeeds if $\mathcal{F}_i^* + \operatorname{lin} s_i$ is closed for all i.*
2. Necessary condition for recovery. *Suppose (x, y) are optimal solutions to (R/P) and (R/D) with zero duality gap, i.e. $\langle c, x\rangle = \langle b, y\rangle$. Then, Algorithm 3 succeeds only if (P) and (D) have solutions with zero duality gap.*

We note the sufficient condition above *always* holds when $\mathcal{K}$ is polyhedral since $\mathcal{F} + \mathcal{M}$ is closed for any subspace $\mathcal{M}$ and face $\mathcal{F}$ of a polyhedral cone. Conversely, $\mathbb{S}^n_+ + \operatorname{lin} S$ is closed only if S is zero or positive definite (which can be shown using essentially the same argument used in [37] to show $\mathbb{S}^n_+ + \operatorname{lin} \mathcal{F}$ is not closed when $\mathcal{F}$ is a proper, non-zero face of $\mathbb{S}^n_+$). Hence, a better sufficient condition for SDP is desired. In the next section, we give a sufficient condition that is also necessary. This condition is specialized to the case $\mathcal{K} = \mathbb{S}^n_+$ when *one* iteration of facial reduction is performed. The restriction to the single iteration case is imposed so that the condition is easy to state, but it can be extended to the multi-iteration case.

Remark 2. *Closure of $\mathcal{K}^* + \operatorname{lin} s$ for $s \in \mathcal{K}^*$ has been studied in other contexts. Borwein and Wolkowicz use this condition to simplify their generalized optimality conditions for convex programs (see Remark 6.2 of [10]). Failure of a related condition, namely closure of $\mathcal{K}^* + \operatorname{lin} \mathcal{F}$ for a face $\mathcal{F}$ of $\mathcal{K}^*$, is used to construct primal-dual pairs with infinite duality gaps in [40].*

5.1 A necessary and sufficient condition for dual recovery

In this section, we give a necessary and sufficient condition (Condition 1) for dual solution recovery that applies when $\mathcal{K} = \mathbb{S}^n_+$ and a single iteration of facial reduction is performed. In this case, the primal-dual pair is given by

$(P - SDP)$:

$$\begin{array}{ll} \text{maximize} & y^T b \\ \text{subject to} & C - \sum_{i=1}^m y_i A_i \succeq 0 \end{array}$$

$(D - SDP)$:

$$\begin{array}{lll} \text{minimize} & C \cdot X & \\ \text{subject to} & A_i \cdot X = b_i & \forall i \in \{1, \ldots, m\} \\ & X \succeq 0, & \end{array}$$

where the primal problem is reproduced from Section 2.4. The reduced primal-dual pair is over a face $\mathcal{F} := \mathbb{S}^n_+ \cap S^\perp$ and its dual cone $\mathcal{F}^* = \overline{\mathbb{S}^n_+ + \operatorname{lin} S}$,

$(R/P - SDP)$:

$$\begin{array}{ll} \text{maximize} & y^T b \\ \text{subject to} & \mathcal{A}(y) = C - \sum_{i=1}^m y_i A_i \\ & U^T \mathcal{A}(y) U \in \mathbb{S}^d_+ \\ & U^T \mathcal{A}(y) V = 0 \\ & V^T \mathcal{A}(y) V = 0 \end{array}$$

$(R/D - SDP)$:

$$\begin{array}{ll} \text{minimize} & C \cdot X \\ \text{subject to} & A_i \cdot X = b_i \qquad \forall i \in \{1, \ldots, m\} \\ & X = (U, V) \begin{pmatrix} W & Z \\ Z^T & R \end{pmatrix} (U, V)^T \\ & W \in \mathbb{S}^d_+, R \in \mathbb{S}^{n-d}, Z \in \mathbb{R}^{d \times n-d}, \end{array}$$

where (U, V) is an invertible matrix that satisfies $S = VV^T$ and $\operatorname{range} U = \operatorname{null} S$. Here, the primal problem is reproduced from Section 4.1, and the dual problem arises from a description of $\mathcal{F}^*$ given by Lemma 1.

Supposing the matrix S is a reducing certificate, i.e. $S \cdot A_i = 0$ and $S \cdot C = 0$, we can construct a solution to the original dual from a solution X to $(R/D - SDP)$ if and only if X is in $\mathbb{S}^n_+ + \operatorname{lin} S$. The following shows this is equivalent to the condition that $\operatorname{null} W \subseteq \operatorname{null} Z^T$.

Lemma 9. *Let (U, V) be an invertible matrix for which $\mathcal{F} := \mathbb{S}^n_+ \cap S^\perp = U\mathbb{S}^d_+ U^T$ and $S = VV^T$. A matrix in the dual cone $\mathcal{F}^* = \overline{\mathbb{S}^n_+ + \operatorname{lin} S}$, i.e. a matrix X of the form*

$$X = (U, V) \begin{pmatrix} W & Z \\ Z^T & R \end{pmatrix} (U, V)^T \qquad \text{for some } W \in \mathbb{S}^d_+, R \in \mathbb{S}^{n-d}, Z \in \mathbb{R}^{d \times n-d}, \tag{12}$$

is in $\mathbb{S}^n_+ + \operatorname{lin} S$ if and only if $\operatorname{null} W \subseteq \operatorname{null} Z^T$.

Proof. For the "only if" direction, suppose X is in $\mathbb{S}^n_+ + \operatorname{lin} S$, i.e. for an $\alpha \in \mathbb{R}$ suppose

$$X + \alpha VV^T = (U, V) \begin{pmatrix} W & Z \\ Z^T & R + \alpha I \end{pmatrix} (U, V)^T \in \mathbb{S}^n_+.$$

Here, membership in $\mathbb{S}^n_+$ holds only if $Z^T(I - WW^\dagger) = 0$, where $(I - WW^\dagger)$ is the orthogonal projector onto $\operatorname{null} W$ (see, e.g. A.5 of [13]). But this implies that $\operatorname{null} W \subseteq \operatorname{null} Z^T$, as desired.

To see the converse direction, suppose X is such that Z and W satisfy $\operatorname{null} W \subseteq \operatorname{null} Z^T$. The result follows by finding α for which $X + \alpha S \succeq 0$. We do this by finding an α_1 and α_2 for which

$$X - VRV^T + \alpha_1 S \succeq 0 \qquad \text{and} \qquad VRV^T + \alpha_2 S \succeq 0.$$

Adding these two inequalities then demonstrates that $X + (\alpha_1 + \alpha_2) S \succeq 0$. To find α_1, we note that

$$X - VRV^T + \alpha_1 S = (U, V) \begin{pmatrix} W & Z \\ Z^T & \alpha_1 I \end{pmatrix} (U, V)^T.$$

Taking a Schur complement, the above is PSD if and only if

$$W - \frac{1}{\alpha_1} ZZ^T \succeq 0.$$

But since $\operatorname{null} W \subseteq \operatorname{null} Z^T$, the matrix ZZ^T is contained in the face $\mathcal{G} = \left\{T \in \mathbb{S}^d_+ : \operatorname{range} T \subseteq \operatorname{range} W\right\}$ where W is in the relative interior of $\mathcal{G}$. This implies existence of $\alpha_1 > 0$ for which $W - \frac{1}{\alpha_1} ZZ^T \in \mathcal{G} \subseteq \mathbb{S}^d_+$, as desired. To find α_2, we note that

$$VRV^T + \alpha_2 S = V(R + \alpha_2 I)V^T,$$

where existence of α_2 for which $R + \alpha_2 I \succeq 0$ is obvious, completing the proof.

$\square$

Using the above characterization of $\mathbb{S}^n_+ + \operatorname{lin} S$ yields our necessary and sufficient condition:

Condition 1. *Suppose $\mathcal{K} = \mathbb{S}^n_+$ and one iteration of facial reduction is performed to $(P - SDP)$. A solution to the dual $(D - SDP)$ can be found using Algorithm 3 if and only if the solution X to the reduced dual $(R/D - SDP)$ satisfies* $\operatorname{null} W \subseteq \operatorname{null} Z^T$.

We give cases where Condition 1 holds and fails in the following example.

5.1.1 Example

Consider the following primal-dual pair:

$$\begin{array}{ll} \text{maximize} & y_3 + 2y_2 \\ \text{subject to} & \\ \mathcal{A}(y) = & \begin{pmatrix} y_1 & y_2 & 0 \\ y_2 & -y_3 & y_2 \\ 0 & y_2 & y_3 \end{pmatrix} \succeq 0 \end{array} \qquad \begin{array}{ll} \text{minimize} & 0 \\ \text{subject to} & x_{33} - x_{22} = -1 \\ & x_{12} + x_{21} + x_{23} + x_{32} = -2 \\ & x_{11} = 0 \\ & X \succeq 0 \end{array}$$

and let $S = VV^T$, with $V = (e_2, e_3)$. Clearly, S is a reducing certificate defining a face $\mathcal{F} := \mathbb{S}^n_+ \cap S^\perp = U\mathbb{S}^1_+ U^T$ for $U = e_1 = (1,0,0)^T$. Rewriting the primal-dual pair over $\mathcal{F}$ and $\mathcal{F}^*$ gives:

$$\begin{array}{ll} \text{maximize} & y_3 + 2y_2 \\ \text{subject to} & \\ \mathcal{A}(y) = & \begin{pmatrix} y_1 & y_2 & 0 \\ y_2 & -y_3 & y_2 \\ 0 & y_2 & y_3 \end{pmatrix} \\ & V^T\mathcal{A}(y)V = 0 \\ & U^T\mathcal{A}(y)V = 0 \\ & U^T\mathcal{A}(y)U \succeq 0 \end{array} \qquad \begin{array}{ll} \text{minimize} & 0 \\ \text{subject to} & x_{33} - x_{22} = -1 \\ & x_{12} + x_{21} + x_{23} + x_{32} = -2 \\ & x_{11} = 0 \\ X & = \begin{pmatrix} w_{11} & z_1 & z_2 \\ z_1 & r_{11} & r_{12} \\ z_2 & r_{12} & r_{22} \end{pmatrix}, w_{11} \geq 0 \end{array}$$

A solution to the dual problem that satisfies Condition 1 is given by:

$$X = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & -1 \end{pmatrix}.$$

To see the condition is satisfied, note $Z = (z_1, z_2) = (0,0)$ and $W = w_{11} = 0$. Hence, $\operatorname{null} Z^T$ contains (indeed, equals) $\operatorname{null} W$. We therefore see that solution recovery succeeds, i.e. for (say) $\alpha = 2$:

$$X + \alpha S = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \alpha & -1 \\ 0 & -1 & \alpha - 1 \end{pmatrix} \succeq 0.$$

Conversely, the following solution fails Condition 1:

$$X = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{13}$$

Here, $Z = (-1, 0)$ and $W = 0$. Hence, $\operatorname{null} Z^T = \{0\}$ does not contain $\operatorname{null} W = \mathbb{R}$ and recovery must fail. In other words, there is no α for which

$$X + \alpha S = \begin{pmatrix} 0 & -1 & 0 \\ -1 & \alpha & 0 \\ 0 & 0 & \alpha - 1 \end{pmatrix} \succeq 0,$$

which is easily seen.

5.1.2 Ensuring successful dual recovery

The condition given in the previous section lets one determine if recovery is possible by a simple null-space computation. Unfortunately, this check must be done *after* the dual problem $(R/D - SDP)$ has been solved. In this section, we give a simple sufficient condition that can be checked *prior* to solving $(R/D - SDP)$. If this condition is satisfied, a modification of the reduced primal-dual pair can be performed to guarantee successful recovery.

The idea is simple: when can one assume $Z = 0$ (and hence ensure Condition 1 holds) without loss of generality? A sufficient condition is given by the following:

Condition 2. *Assume $(R/D - SDP)$ has a solution. If $V^T\mathcal{A}(y)V$ implies $V^T\mathcal{A}(y)U = 0$, i.e. if*

$$\left\{y \in \mathbb{R}^m : V^T\mathcal{A}(y)V = 0\right\} = \left\{y \in \mathbb{R}^m : V^T\mathcal{A}(y)V = 0, V^T\mathcal{A}(y)U = 0\right\},$$

then $(R/D - SDP)$ has a solution with $Z = 0$.

A proof of the above arises by noting Z is a matrix of Lagrange multipliers for the linear equations $V^T\mathcal{A}(y)U = 0$. Hence, if these equations are implied by $V^T\mathcal{A}(y)V = 0$, i.e. each entry of $V^T\mathcal{A}(y)U$ is a linear combination of entries of $V^T\mathcal{A}(y)V$, then there is a map (defined by the specific linear combinations) taking any solution of $(R/D - SDP)$ to a solution with $Z = 0$.

Note when Condition 2 holds, one can solve a modified reduced-primal dual pair that fixes Z to zero and omits the equations $V^T\mathcal{A}(y)U = 0$:

$(R/P - SDP - 2)$:

$$\begin{array}{ll} \text{maximize} & y^T b \\ \text{subject to} & \mathcal{A}(y) = C - \sum_{i=1}^m y_i A_i \\ & U^T\mathcal{A}(y)U \in \mathbb{S}^d_+ \\ & V^T\mathcal{A}(y)V = 0 \end{array}$$

$(R/D - SDP - 2)$:

$$\begin{array}{ll} \text{minimize} & C \cdot X \\ \text{subject to} & A_i \cdot X = b_i \qquad \forall i \in \{1, \ldots, m\} \\ & X = (U, V)\begin{pmatrix} W & 0 \\ 0 & R \end{pmatrix}(U, V)^T \\ & W \in \mathbb{S}^d_+, R \in \mathbb{S}^{n-d}. \end{array}$$

Here, any solution to $(R/P - SDP - 2)$ solves the original primal $(P - SDP)$ and—assuming one iteration of facial reduction was performed—any solution to $(R/D - SDP - 2)$ can be used to recover a solution to $(D - SDP)$ since Condition 1 is satisfied by construction.

5.1.3 Strong duality is not sufficient for dual recovery

An additional observation can be made about the example of Section 5.1.1. As we observed in Lemma 8, zero duality gap between the original primal-dual pair (P) and (D) is a necessary condition for recovery to succeed when the reduced primal-dual pair (R/D) and (R/D) has zero duality gap. The example of Section 5.1.1 shows this is *not* a sufficient condition when $\mathcal{K} = \mathbb{S}^n_+$. Here, both the original and the reduced primal-dual pairs have zero duality gap yet successful recovery depends on the specific solution found for the reduced dual $(R/D - SDP)$. This is summarized below:

Corollary 4. *The dual solution recovery procedure of Algorithm 3 can fail even if both the original primal-dual pair (P) and (D) and the reduced primal-dual pair (R/P) and (R/D) have zero duality gap.*

5.2 Recovering solutions to an extended dual

We close this section by discussing recovery for an alternative dual program intimately related to facial reduction—a so-called *extended dual* [31]. For SDP, this dual is a slight variant of the Ramana dual [36], which was related to facial reduction in [37].

A solution to an extended dual carries the same information as a solution to the reduced dual (R/D) and a sequence of reducing certificates used to identify a face. However, such a solution allows one to certify optimality of the primal problem (P) without retracing the steps of the facial reduction algorithm (to verify validity of each reducing certificate)—one simply checks that a solution to an extended dual and a candidate solution to (P) have zero duality gap.

Extended duals can be defined for cones $\mathcal{K}$ that are *nice* [31], but we will limit discussion to the case when $\mathcal{K} = \mathbb{S}^n_+$. The extended dual considered is based on three key facts.

Lemma 10. *The following statements are true:*

1. *For any face $\mathcal{F}$ of $\mathbb{S}^n_+$, $\mathcal{F}^* = \mathbb{S}^n_+ + \mathcal{F}^\perp$.*
2. *If $\mathcal{F} = \mathbb{S}^n_+ \cap S^\perp$ for $S \in \mathbb{S}^n_+$, then*

$$\mathcal{F}^\perp = \left\{ W + W^T : \begin{pmatrix} S & W \\ W^T & \alpha I \end{pmatrix} \succeq 0 \text{ for some } \alpha \in \mathbb{R} \right\}.$$

3. *Let $\mathcal{F}_0 := \mathbb{S}^n_+$ and consider the chain of faces defined by matrices S_i*

$$\mathcal{F}_{i+1} := \mathcal{F}_i \cap S_i^\perp,$$

where S_i is in $\mathcal{F}_i^*$, i.e. $S_i = \bar{S}_i + V_i$ for $\bar{S}_i \in \mathbb{S}^n_+$ and $V_i \in \mathcal{F}_i^\perp$. The following relationship holds:

$$\mathcal{F}_{i+1} = \mathbb{S}^n_+ \cap (\sum_{j=0}^{i} \bar{S}_j)^\perp.$$

Proof. The first statement holds because $\mathbb{S}^n_+$ is a *nice* cone [31]. The other statements are shown by Proposition 1 and Theorem 3 of [31]. ☐

Using these facts, the extended dual considered simultaneously identifies a chain of faces $\mathcal{F}_1, \ldots, \mathcal{F}_N$ (where N can be chosen to equal the length of the longest chain of faces of $\mathbb{S}^n_+$) and a solution $X \in \mathcal{F}_N^*$ to the reduced dual $(R/D{-}SDP)$ formulated over $\mathcal{F}_N^*$. It is given below as an optimization problem over $X, \bar{X}, W_N, S_i, \bar{S}_i, W_i, \alpha_i$:

$$
\begin{array}{llr l}
(EXT/D-SDP) & & & \\
\text{minimize} & & C \cdot X & \\
\text{subject to} & & A_j \cdot X = b_j & \\
& & C \cdot S_i = 0,\ A_j \cdot S_i = 0 & \text{(i.e. } S_i^\perp \text{ contains } \mathcal{A}) \\
& & X = \bar{X} + W_N + W_N^T & \text{(i.e. } X \in \mathbb{S}^n_+ + \mathcal{F}_N^\perp = \mathcal{F}_N^*) \\
& & S_i = \bar{S}_i + W_i + W_i^T & \text{(i.e. } S_i \in \mathbb{S}^n_+ + \mathcal{F}_i^\perp = \mathcal{F}_i^*) \\
& & \begin{pmatrix} \sum_{j=0}^{i} \bar{S}_j & W_{i+1} \\ W_{i+1}^T & \alpha_i I \end{pmatrix} \succeq 0 & \text{(i.e. } W_{i+1} + W_{i+1}^T \in \mathcal{F}_{i+1}^\perp) \\
& & \bar{S}_i \succeq 0, \bar{X} \succeq 0, W_0 = 0, &
\end{array}
$$

where i ranges from 0 to $N-1$ and j ranges from 1 to m (indexing m linear equations $A_j \cdot X = b_j$).

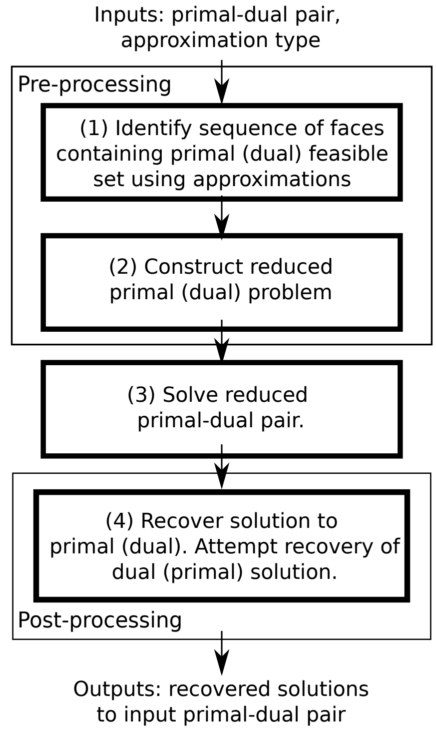

Figure 2: Flow of MATLAB implementation

Recovering a solution Suppose $\mathcal{F}_i = U_i \mathbb{S}_+^{d_i} U_i^T$ for $i = 0, \ldots, N$ is a sequence of faces identified by an SDP facial reduction procedure (e.g. Algorithm 2, with or without the modifications of Section 3) suitably padded so that the length of the sequence is N, i.e. $\mathcal{F}_0, \ldots, \mathcal{F}_M = \mathbb{S}_+^n$ for some $M < N$. Let $S_i \in \mathcal{F}_i^*$ be the corresponding sequence of reducing certificates (similarly padded with zeros) and let X be a solution to $(R/D - SDP)$. One can construct a feasible point to $(EXT/D - SDP)$ by decomposing S_i (and similarly X) into the form $S_i = \bar{S}_i + W_i + W_i^T$, for $\bar{S}_i \in \mathbb{S}_+^n$ and $W_i + W_i^T \in \mathcal{F}_i^\perp$. Supposing U_i has orthonormal columns, this can be done by taking:

$$\begin{array}{lll} \bar{S}_i = U_i U_i^T S_i U_i U_i^T & W_i = \frac{1}{2}(S - \bar{S}_i) & \forall i \in \{0, \ldots, N-1\} \\ \bar{X} = U_N U_N^T X U_N U_N^T & W_N = \frac{1}{2}(X - \bar{X}). & \end{array}$$

One can then pick α_i (individually) until the relevant semidefinite constraint is satisfied. The feasible point produced by this procedure is optimal if the reduced primal-dual pair over $\mathcal{F}_N$ and $\mathcal{F}_N^*$ has no duality gap. This of course occurs if the reduced primal problem over $\mathcal{F}_N$ is strictly feasible (i.e. the unmodified version of Algorithm 2 is run to completion).

6 Implementation

The discussed techniques have been implemented as a suite of MATLAB scripts we dub `frlib`, available at at `www.mit.edu/~fperment`. The basic work flow is depicted in Figure 2. The implemented code takes as input a primal-dual SDP *pair* and can reduce (using suitable variants of Algorithm 2) either the primal problem or the dual. This is an important feature since either the primal or the dual may model the problem of interest.

6.1 Input formats

The implementation takes in SeDuMi-formatted inputs `A,b,c,K`, where `A,b,c`, define the subspace constraint and objective function and `K` specifies the sizes of the semidefinite constraints [39]. Conventionally, the primal problem described by `A,b,c,K` refers to an SDP defined by equations $A_i \cdot X = b_i$. Similarly, the dual problem described by `A,b,c,K` refers to an SDP defined by generators $C - \sum_i y_i A_i$. While our implementation and the following discussion follow this convention, the *opposite* convention was used in previous sections (e.g. $(SDP-P)$ and $(SDP-D)$ in Section 5.1).

6.2 Reduction of the primal problem

Given `A,b,c,K`; the following syntax is used to reduce the primal problem, solve the reduced primal-dual pair, and recover solutions to the original primal-dual pair via our implementation:

```
prg = frlibPrg(A,b,c,K);
prgR = prg.ReducePrimal('d');
[x_reduced,y_reduced] = sedumi(prgR.A, prgR.b, prgR.c, prgR.K);
[x,y,dual_recov_success] = prgR.Recover(x_reduced,y_reduced);
```

The call to `prg.ReducePrimal` reduces the primal problem using diagonal (`'d'`) approximations by executing a variant of Algorithm 1. To find reducing certificates, it solves a series of LPs (defined by the diagonal approximation) that can be solved using a handful of supported solvers. The returned object `prgR` has member variables

`prgR.A, prgR.b, prgR.c, prgR.K,`

which describe the reduced primal-dual pair. For a single semidefinite constraint, this reduced primal-dual pair is given by:

$$\begin{array}{lll} \text{minimize} & C \cdot U\hat{X}U^T & \\ \text{subject to} & A_i \cdot U\hat{X}U^T = b_i \ \ \forall i \in \{1,\dots,m\} \\ & \hat{X} \in \mathbb{S}^d_+ \end{array} \qquad \begin{array}{ll} \text{maximize} & y^T b \\ \text{subject to} & U^T(C - \sum_{i=1}^m y_i A_i)U \in \mathbb{S}^d_+, \\ & \end{array}$$

where $U\mathbb{S}^d_+U^T$ is a face identified by `prg.ReducePrimal`. The reduced primal and its dual are solved by calling SeDuMi.

The primal solution `x_reduced` returned by SeDuMi represents an optimal $\hat{X}$. The function `prgR.Recover` computes from $\hat{X}$ a solution $U\hat{X}U^T$ to the original primal problem. It then attempts to find a solution to the original dual using a variant of the recovery procedure described in Section 5 (Algorithm 3). The flag `dual_recov_success` indicates success of this recovery procedure.

6.3 Reduction of the dual problem

The above syntax can be modified to reduce the dual problem described by `A,b,c,K`. This is done replacing the relevant line above with:

```
prgR = prg.ReduceDual('d');
```

As above, the object `prgR` contains a description of the primal-dual pair which, for a single semidefinite constraint, is given by the SDPs $(R/P-SDP)$ $(R/D-SDP)$ in Section 5.1 (where, recalling our earlier convention, the label of "primal" and "dual" is reversed). With `prgR` created in this manner, a call to `prgR.Recover` (though syntactically identical) now returns a solution to the original *dual* and attempts to recover a solution to the original *primal* using Algorithm 3. In other words, a call of the form

```
[x,y,prim_recov_success] = prgR.Recover(x_reduced,y_reduced);
```

returns a solution `y` to the original dual problem and attempts to recover a solution `x` to the original primal problem. The flag `prim_recov_success` indicates successful recovery of `x`.

6.4 Solution recovery

As suggested by the flags `prim_recov_success` and `dual_recov_success` in the preceding examples, solution recovery is only guaranteed for the problem that is reduced, i.e. if the primal (resp. dual) is reduced, recovery of the original dual (resp. primal) may fail for reasons discussed in Section 5.1. Thus, it is important to reduce the primal only if it is the problem of interest, and similarly for the dual.

6.5 Other approximations

The implemented code currently supports non-negative diagonal ($\mathcal{D}^d$) and diagonally-dominant ($\mathcal{DD}^d$) approximations. While the preceding examples illustrate the use of diagonal approximations, diagonally-dominant approximations can be used via a call of the form:

```
prgR = prg.ReducePrimal('dd');
```

or

```
prgR = prg.ReduceDual('dd');
```

6.6 Interface to commercial solvers

As mentioned, the functions `prg.ReducePrimal` and `prg.ReduceDual` find reducing certificates by solving linear programs. To solve these LPs, commercial solvers can be used. Currently, LP solvers of MOSEK [28], Gurobi [29] and `linprog`—the LP solver available in the MATLAB optimization toolbox [14]—are supported.

7 Examples

This section gives larger examples that illustrate the practical utility of our method. The size n of the cone and the dimension r of the subspace that together define the feasible set is given for each example, before and after reductions. Also given is the total time $t_{LPs/SOCPs}$ (in seconds) spent solving LPs/SOCPs for reducing certificates. These solve times are reported for an Intel(R) Core(TM) i7-2600K CPU @ 3.40GHz machine with 16 gigabytes of RAM using the solver MOSEK.

For each example, the same type of approximation (e.g. diagonal or diagonally-dominant) is used at every iteration. Note that many examples are over products of cones, e.g. $\mathcal{K} = \mathbb{S}^{n_1} \times \mathbb{S}^{n_2} \cdots \times \mathbb{S}^{n_k}$. In these cases, we use the same type of approximation for each cone $\mathbb{S}^{n_i}$. This leads to the following abuse of notation: for a given example, we will let $\mathcal{C}(\mathbb{W})^* = \mathcal{D}^d$ (for instance) denote the use of diagonal approximations at every iteration for every semidefinite cone.

As we see from the examples, problem sizes are significantly reduced in time reasonable for a pre-processor. In addition to being more easily solved, the reduced problems can also lead to higher quality solutions, as indicated by Example 7.7.

Example	Original $n; r$	Reduced $n; r$	t_{LPs}
`long_only`	$n_{1:100} = 91,\ n_{101:200} = 30; 59095$	$n_{1:100} = 61,\ n_{101:200} = 30; 56095$	0.33
`unconstrained`	$n_{1:100} = 121,\ n_{101:200} = 30; 62095$	$n_{1:100} = 61,\ n_{101:200} = 30; 56095$	0.71
`sector_neutral`	$n_{1:100} = 121,\ n_{101:200} = 30; 62392$	$n_{1:100} = 61,\ n_{101:200} = 30; 56392$	0.70
`leverage_limit`	$n_{1:100} = 151,\ n_{101:200} = 30; 68195$	$n_{1:100} = 61,\ n_{101:200} = 30; 59195$	1.2

Table 2: Complexity parameters for the multi-period investment problems from [12] described in Section 7.1. The feasible set of each SDP is an r-dimensional subspace intersected with a cone defined by 200 semidefinite constraints of size n_i. To formulate each reduced SDP, a face was identified by solving LPs defined by diagonal approximations ($\mathcal{D}^d$). These LPs took (in total) t_{LPs} seconds to solve.

7.1 Lower bounds for optimal multi-period investment

Our first example arises from SDP-based lower bounds of optimal multi-period investment strategies. The strategies and specific SDP formulations are given in [12]. For each strategy, an SDP produces a quadratic lower bound on the value function arising in the dynamic programming solution to the underlying optimization problem. These bounds are produced using the S-procedure, an SDP-based method for showing emptiness of sets defined by quadratic polynomials (see, e.g., [11]). Simplifications of each SDP are summarized in Table 2. Scripts that generate the SDPs are found here (and require the package CVX [22]):

`www.stanford.edu/~boyd/papers/matlab/port_opt_bound/port_opt_code.tgz`

7.2 Copositivity of quadratic forms

Our next example pertains to SDPs that demonstrate *copositivity* of certain quadratic forms. A quadratic form $x^T Jx$ is copositive if and only if $x^T Jx \geq 0$ for all x in the non-negative orthant. Deciding copositivity is NP-hard, but a sufficient condition can be checked using sum-of-squares techniques and semidefinite programming, as we now illustrate.

The Horn form An example of a copositive polynomial is the Horn form $f(x) := x^T Jx$, where

$$J = \begin{pmatrix} 1 & -1 & 1 & 1 & -1 \\ -1 & 1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & 1 & -1 \\ -1 & 1 & 1 & -1 & 1 \end{pmatrix}, \qquad x = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 \end{pmatrix}^T.$$

This polynomial, originally introduced by A. Horn, appeared previously in [20] [35]. To see how copositivity can be demonstrated using SDP, first note copositivity of $f(x)$ is equivalent to global non-negativity of $f(z_1^2, z_2^2, z_3^2, z_4^2, z_5^2)$, where we have substituted each variable x_i with the square of a new indeterminate z_i^2. Next, note global non-negativity of the latter polynomial can be demonstrated by showing

$$g(z) = \left(\sum_{i=1}^{5} z_i^2\right) f(z_1^2, z_2^2, z_3^2, z_4^2, z_5^2)$$

Polynomial	Original $n; r$	Reduced $n; r$	t_{LPs}
$g(z_1, z_2, 0, 0, 0)$	$4; 3$	$2; 0$	.012
$g(z_1, z_2, z_3, 0, 0)$	$10; 27$	$6; 5$	.012
$g(z_1, z_2, z_3, z_4, 0)$	$20; 126$	$14; 45$	.019
$g(z_1, z_2, z_3, z_4, z_5)$	$35; 420$	$25; 165$	.047

(a) Horn form

m	Original $n; r$	Reduced $n; r$	t_{LPs}
2	$120; 5544$	$96; 3132$	.58
3	$286; 33033$	$242; 21879$	4.3
4	$560; 129948$	$490; 494143$	24
5	$969; 395352$	$867; 303399$	83

(b) Generalized Horn forms

Table 3: Complexity parameters for the copositivity examples of Section 7.2 before and after facial reduction. The feasible set of each SDP is an r-dimensional subspace intersected with the cone $\mathbb{S}^n_+$. To formulate each reduced SDP, a face was identified by solving LPs defined by diagonally-dominant approximations ($\mathcal{DD}^d$). These LPs took (in total) t_{LPs} seconds to solve.

is a sum-of-squares, which is equivalent to feasibility of a particular SDP.

We construct SDPs that test if the polynomial $g(z_1, z_2, z_3, z_4, z_5)$ and the related polynomials $g(z_1, z_2, 0, 0, 0)$, $g(z_1, z_2, z_3, 0, 0)$, $g(z_1, z_2, z_3, z_4, 0)$ and $g(z_1, z_2, z_3, z_4, z_5)$ are sums-of-squares. Letting p denote the number of non-zero indeterminates, each SDP is formulated using the set of $\binom{p+2}{3}$ degree-three monomials in p variables (see Chapter 3 of [8] for details on this formulation). Reductions for each SDP are summarized in Table 3 and were performed using diagonally-dominant ($\mathcal{DD}^d$) approximations.

Generalized Horn forms The Horn form $f(x)$ generalizes to a family of copositive forms in $n = 3m + 2$ variables ($m \geq 1$):

$$B(x; m) = \left(\sum_{i=1}^{3m+2} x_i \right)^2 - 2 \sum_{i=1}^{3m+2} x_i \sum_{j=0}^{m} x_{i+3j+1},$$

where we let the subscript for the indeterminate x wrap cyclically, i.e. $x_{r+n} = x_r$. This family was studied in [6], and the Horn form corresponds to the case $m = 1$. Using the same construction as for the Horn form, we formulate SDPs that demonstrate copositivity of $B(x; m)$ for each $m \in \{2, \ldots, 5\}$. We then reduce these SDPs using diagonally-dominant approximations ($\mathcal{DD}^d$). Results also appear in Table 3.

7.3 Lower bounds on completely positive rank

A matrix $A \in \mathbb{S}^n$ is *completely positive* (CP) if there exist r non-negative vectors $v_i \in \mathbb{R}^n$ for which

$$A = \sum_{i=1}^{r} v_i v_i^T. \tag{14}$$

The completely positive rank of A, denoted $\operatorname{rank}_{\text{cp}} A$, is the smallest r for which A admits the decomposition (14). It follows trivially that

$$\operatorname{rank} A \leq \operatorname{rank}_{\text{cp}} A.$$

In [21], Fawzi and the second author give an SDP formulation that improves this lower bound for a fixed matrix A. This bound, denoted $\tau_{cp}^{sos}(A)$ in [21], equals the optimal value of the following

Matrix	Original $n; r$	Reduced $n; r$	t_{LPs}
Z	$(9, 10, 9); 37$	$(7, 8, 9); 20$	.0084
$Z \otimes Z$	$(81, 82, 81); 2026$	$(49, 50, 81); 464$	.016
$Z \otimes Z \otimes Z$	$(729, 730, 729); 142885$	$(343, 344, 729); 13262$	.58

Table 4: Complexity parameters for SDPs that lower bound cp-rank for different matrices (where Z is defined in Section 7.3). The feasible set of each SDP is an r-dimensional subspace intersected with the cone $\mathbb{R}^{n_1}_+ \times \mathbb{S}^{n_2}_+ \times \mathbb{S}^{n_3}_+$. To formulate each reduced SDP, a face was identified by solving LPs using diagonal approximations ($\mathcal{D}^d$). These LPs took (in total) t_{LPs} seconds to solve.

semidefinite program:

$$\begin{array}{ll} \text{minimize } t & \\ \text{subject to} & \\ & \begin{pmatrix} t & \operatorname{vect} A^T \\ \operatorname{vect} A & X \end{pmatrix} \succeq 0 \\ & X_{ij,ij} \leq A_{ij}^2 \quad \forall i, j \in \{1, \ldots, n\} \\ & X \preceq A \otimes A \\ & X_{ij,kl} = X_{il,jk} \quad \forall (1,1) \leq (i,j) < (k,l) \leq (n,n), \end{array}$$

where $A \otimes A$ denotes the *Kronecker product* and $\operatorname{vect} A$ denotes the $n^2 \times 1$ vector obtained by stacking the columns of A. Here, the double subscript ij indexes the n^2 rows (or columns) of X and the inequalities on (i, j) hold iff they hold element-wise (see [21] for further clarification on this notation).

In this example, we formulate SDPs as above for computing $\tau_{cp}^{sos}(Z)$, $\tau_{cp}^{sos}(Z \otimes Z)$, and $\tau_{cp}^{sos}(Z \otimes Z \otimes Z)$, where Z is the completely positive matrix:

$$Z = \begin{pmatrix} 4 & 0 & 1 \\ 0 & 4 & 1 \\ 1 & 1 & 3 \end{pmatrix}.$$

Notice that since Z is CP, the Kronecker products $Z \otimes Z$ and $Z \otimes Z \otimes Z$ are CP (using the fact that $A \otimes B$ is CP when A and B are CP [7]). Also notice that since Z contains zeros, the constraint $X_{ij,ij} \leq Z_{ij}^2$ implies that X has rows and columns identically zero; in other words, because Z has elements equal to zero, the SDP for computing $\tau_{cp}^{sos}(Z)$ cannot have a strictly feasible solution.

To reduce the formulated SDPs, we first observe that each is actually a cone program over $\mathbb{R}^{n_1}_+ \times \mathbb{S}^{n_2}_+ \times \mathbb{S}^{n_3}_+$, i.e. each SDP has a mix of linear inequalities and semidefinite constraints. To find reductions, we first treat the linear equalities as a semidefinite constraint on a diagonal matrix. We then perform reductions using diagonal approximations ($\mathcal{D}^d$). Results are reported in Table 4.

7.4 Lyapunov Analysis of a Hybrid Dynamical System

The next example arises from SDP-based stability analysis of a *rimless wheel*, a hybrid dynamical system and simple model for walking robots studied in [33] by Posa, Tobenkin, and Tedrake. The SDP includes several coupled semidefinite constraints that impose Lyapunov-like stability conditions accounting for Coulomb friction and the contact dynamics of the rimless wheel. Table 5 shows solve time and problem size reduction achieved using diagonal ($\mathcal{D}^d$) and diagonally-dominant ($\mathcal{DD}^d$) approximations.

$\mathcal{C}(\mathbb{W})^*$	Original $n; r$	Reduced $n; r$	t_{LPs}
$\mathcal{D}^d$	$(6, 108, 11 \times 10)$; 4334	$(6, 56, 11, 1, 1, 0, 11, 1, 1, 0, 11, 11)$; 1138	.05
$\mathcal{DD}^d$	$(6, 108, 11 \times 10)$; 4334	$(6, 34, 8, 1, 1, 0, 8, 1, 1, 0, 9, 7)$; 452	.82

Table 5: Complexity parameters for the SDPs of [33] arising in Lyapunov analysis of a hybrid dynamical system (Section 7.4). The feasible set of each SDP is an r-dimensional subspace intersected with the cone $\mathbb{S}^{n_1}_+ \times \mathbb{S}^{n_2}_+ \ldots \times \mathbb{S}^{n_{12}}_+$. To formulate each reduced SDP, a face was identified by solving LPs defined by $\mathcal{C}(\mathbb{W})^*$. These LPs took (in total) t_{LPs} seconds to solve.

7.5 Multi-affine polynomials, matroids, and the half-plane property

A multivariate polynomial $f(z) : \mathbb{C}^n \to \mathbb{C}$ has the *half-plane* property if it is non-zero when each variable z_i has positive real part. A polynomial is *multi-affine* if each indeterminate is raised to at most the first power. As proven in [19], if a multi-affine, homogeneous polynomial with unit coefficients has the half-plane property, it is the *basis generating polynomial* of a *matroid*. In this section, we reduce SDPs that arise in the study of the converse question: given a matroid, does its basis generating polynomial have the half-plane property? Or more precisely, given a rank-r matroid M (over the ground-set $\{1, \ldots, n\}$) with set of bases $B(M)$, does the multi-affine, degree-r polynomial

$$f_M(z_1, \ldots, z_n) := \sum_{\substack{\{i_1, i_2, \ldots, i_r\} \\ \in B(M)}} z_{i_1} z_{i_2} \cdots z_{i_r} \tag{15}$$

have the half-plane property?

The role of polynomial non-negativity This converse question is related to global non-negativity of so-called *Rayleigh differences* of $f_M(z)$, which are polynomials over $\mathbb{R}^n$ defined for each $\{i, j\} \subset \{1, \ldots, n\}$ as follows:

$$\Delta_{ij} f_M(x) := \frac{\partial f_M}{\partial z_i}(x) \frac{\partial f_M}{\partial z_j}(x) - \frac{\partial^2 f_M}{\partial z_i \partial z_j}(x) \cdot f_M(x).$$

A theorem of Brändén [15] states $f_M(z)$ has the half-plane property if and only if all of $\binom{n}{2}$ Rayleigh differences are globally non-negative, i.e. $\Delta_{ij} f_M(x) \geq 0$ for all $x \in \mathbb{R}^n$. An equivalent criterion, stated in terms of global non-negativity of a single Rayleigh difference (and so-called *contractions* and *deletions* of M), appears in [41].

The role of semidefinite programming Since semidefinite programming can demonstrate a given polynomial is a sum-of-squares, it is a natural tool for proving a given Rayleigh difference $\Delta_{ij} f_M(x)$ is globally non-negative. In this section, we formulate and then apply our reduction technique to SDPs that test the sum-of-squares condition for various $\Delta_{ij} f_M(x)$ and various matroids M. As is standard, the SDPs are formulated using the set of monomial exponents in $\frac{1}{2}\mathcal{N}(\Delta_{ij} f_M) \cap \mathbb{N}^n$, where $\mathcal{N}(\Delta_{ij} f_M)$ denotes the *Newton polytope* of $\Delta_{ij} f_M$ (see Chapter 3 of [8] for details on this formulation).

M	$\{i,j\}$	$n;r$ Original	$n;r$ Reduced	t_{LPs}
$\mathcal{F}_7^{-4}$	{1,2}	(8;5)	(5;1)	.0087
$\mathcal{W}^{3+}$	{1,2}	(8;5)	(3;0)	.010
$\mathcal{W}^3+e$	{1,2}	(9;7)	(5;0)	.011
$\mathcal{P}_7'$	{1,2}	(8;4)	(4;0)	.011
$n\mathcal{P}\setminus 1$	{2,4}	(12;14)	(6;0)	.013
$n\mathcal{P}\setminus 9$	{1,2}	(12;14)	(5;0)	.011
$\mathcal{V}_8$	{1,2}	(16;33)	(13;17)	.014
$\mathcal{V}_{10}$	{3,4}	(52;657)	(41;327)	0.084

Table 6: Complexity parameters for different matroids discussed in Section 7.5.1 and Section 7.5.2. The feasible set of each SDP is an r-dimensional subspace intersected with the cone $\mathbb{S}^n_+$. To formulate each reduced SDP, a face was identified by solving LPs defined by diagonally-dominant approximations ($\mathcal{DD}^d$). These LPs took (in total) t_{LPs} seconds to solve.

7.5.1 Various matroids with the half-plane property

The first set of SDPs we consider arises from matroids studied in [41] by Wagner and Wei. There, it is demonstrated that $\Delta_{ij}f_M$ (for specific $\{i,j\}$) is a sum-of-squares for matroids M they denote $\mathcal{F}_7^{-4}$, $\mathcal{W}^{3+}$, $\mathcal{W}^3+e$, $\mathcal{P}_7'$, $n\mathcal{P}\setminus 1$, $n\mathcal{P}\setminus 9$, and $\mathcal{V}_8$. (We refer the reader to [41] for definitions of these matroids and the explicit polynomials $\Delta_{ij}f_M$.) While Wagner and Wei demonstrate each sum-of-squares condition via ad-hoc construction, it is natural to ask if SDPs that demonstrate these conditions can be reduced via our technique. To this end, we formulate these SDPs and apply reductions using diagonally-dominant ($\mathcal{DD}^d$) approximations. Results are given in Table 6.

Notice from Table 6 that for matroids $\mathcal{W}^{3+}$, $\mathcal{W}^3+e$, $\mathcal{P}_7'$, $n\mathcal{P}\setminus 1$ and $n\mathcal{P}\setminus 9$, the reduced SDP is described by a zero-dimensional affine subspace. In other words, the SDP demonstrating the sum-of-squares condition has a feasible set containing a single point.

7.5.2 Extended Vámos matroid

Another SDP comes from a matroid studied by Burton, Vinzant, and Youm in [17]. There, the authors use semidefinite programming to show $\Delta_{ij}f_{\mathcal{V}_{10}}$ is a sum-of-squares for a specific $\{i,j\}$, where $\mathcal{V}_{10}$ denotes the *extended Vámos matroid* defined over the ground set $\{1,\dots,10\}$. The bases of $\mathcal{V}_{10}$ are all cardinality-four subsets of $\{1,\dots,10\}$ excluding

$$\{1,2,6,7\},\{1,3,6,8\},\{1,4,6,9\},\{1,5,6,10\},\{2,3,7,8\},\{3,4,8,9\}, \text{ and } \{4,5,9,10\}.$$

From these bases, we construct $f_{\mathcal{V}_{10}}$ via (15) and formulate an SDP demonstrating $\Delta_{34}f_{\mathcal{V}_{10}}$ is a sum-of-squares (as was done in [17]). Reductions are applied using diagonally-dominant ($\mathcal{DD}^d$) approximations. Results are also shown in Table 6.

7.6 Facial Reduction Benchmark Problems

In [18], Cheung, Schurr, and Wolkowicz developed a facial reduction procedure for identifying faces in a numerically stable manner. They also created a set of benchmark problems for testing their method. These problem instances are available at the URL below:

`http://orion.math.uwaterloo.ca/~ hwolkowi/henry/reports/SDPinstances.tar`.

Example	Original Primal $n; r$	Reduced Primal $n; r$	t_{LPs}	Original Dual $n; r$	Reduced Dual $n; r$	t_{LPs}
`Example1`	$3; 4$	$2; 2$	.0089	$3; 2$	$1; 1$	.012
`Example2`	$3; 4$	$2; 2$	.0079	$3; 2$	$2; 1$	.0080
`Example3`	$3; 2$	$2; 2$	.0081	$3; 4$	$2; 2$	.0081
`Example4`	$3; 3$	$1; 0$	.012	$3; 3$	$1; 1$	.012
`Example5`	$10; 50$	$10; 50$	.007	$10; 5$	$10; 5$	.006
`Example6`	$8; 28$	$5; 11$	.0084	$8; 8$	$4; 4$	.0084
`Example7`	$5; 12$	$4; 8$	.0082	$5; 3$	$1; 1$	.012
`Example9a`	$100; 4950$	$1; 0$	.10	$100; 100$	$1; 1$	2.0
`Example9b`	$20; 190$	$1; 0$	.016	$20; 20$	$1; 1$	.096

Table 7: Complexity parameters for the primal-dual SDP pairs given in [18]. The feasible set of each SDP is an r-dimensional subspace intersected with the cone $\mathbb{S}^n_+$. To formulate each reduced SDP, a face was identified by solving LPs over diagonally-dominant approximations ($\mathcal{DD}^d$). These LPs took (in total) t_{LPs} seconds to solve.

Each problem is a primal-dual pair hand-crafted so that both the primal and dual have no strictly feasible solution. We apply our technique to each primal problem and each dual problem individually, using diagonally-dominant ($\mathcal{DD}^d$) approximations. Results are shown in Table 7.

7.7 Difficult SDPs arising in polynomial non-negativity

In [42] and [44], Waki et al. study two sets of SDPs that are difficult to solve. For one set of SDPs, SeDuMi fails to find certificates of infeasibility [42]. For the other set, SeDuMi reports an incorrect optimal value [44]. The sets of SDPs are available at:

`https://sites.google.com/site/hayatowaki/Home/difficult-sdp-problems`.

It turns out for each primal-dual pair in these sets, the problem defined by equations $A_i \cdot X = b_i$ is not strictly feasible. We apply our technique to both sets of SDPs using diagonal approximations $\mathcal{D}^d$ and arrive at SDPs that are more easily solved. In particular, certificates of infeasibility are found for the SDPs in [42] and correct optimal values are found for the SDPs in [44]. Problem size reductions are shown in Table 8 and Table 9.

7.8 DIMACS Controller Design Problems

Our final examples are the controller design problems `hinf12` and `hinf13` of the DIMACS library [32]—which evidently are SDPs in the library with no strictly feasible solution. Results are shown in Table 10. For `hinf13`, we use scaled diagonally-dominant ($\mathcal{SDD}^d$) matrices for our approximation. For this non-polyhedral approximation, reducing certificates are found by solving SOCPs.

8 Conclusion

We presented a general technique for facial reduction that utilizes approximations of the positive semidefinite cone. The technique is effective on examples arising in practice and for simple approximation is a practical pre-processing routine for SDP solvers. An implementation has been

Example	$n; r$ Original	$n; r$ Reduced	t_{LPs}
`CompactDim2R1`	3;4	1;1	.0081
`CompactDim2R2`	(6,3,3,3); 25	(1,0,1,1); 1	.026
`CompactDim2R3`	(10,6,6,6); 91	(1,0,1,1); 1	.042
`CompactDim2R4`	(15,10,10,10); 241	(1,0,1,1); 1	.060
`CompactDim2R5`	(21,15,15,15); 526	(1,0,1,1); 1	.083
`CompactDim2R6`	(28,21,21,21); 1009	(1,0,1,1); 1	.11
`CompactDim2R7`	(36,28,28,28); 1765	(1,0,1,1); 1	.14
`CompactDim2R8`	(45,36,36,36); 2881	(1,0,1,1); 1	.19
`CompactDim2R9`	(55,45,45,45); 4456	(1,0,1,1); 1	.25
`CompactDim2R10`	(66,55,55,55); 6601	(1,0,1,1); 1	.32

Table 8: Complexity parameters for weakly-infeasible SDPs studied in [42]. The feasible set of each SDP is an r-dimensional subspace intersected with the cone $\mathbb{S}^n_+$. To formulate each reduced SDP, a face was identified by solving LPs defined by diagonal approximations ($\mathcal{D}^d$). These LPs took (in total) t_{LPs} seconds to solve.

Example	$n; r$ Original	$n; r$ Reduced	t_{LPs}	Optimal Value Reduced
`unboundDim1R2`	(3,2,2); 8	(1,1,0); 1	.017	1.080478e-13
`unboundDim1R3`	(4,3,3); 16	(1,1,0); 1	.026	1.080478e-13
`unboundDim1R4`	(5,4,4); 27	(1,1,0); 1	.033	1.080478e-13
`unboundDim1R5`	(6,5,5); 41	(1,1,0); 1	.043	1.080478e-13
`unboundDim1R6`	(7,6,6); 58	(1,1,0); 1	.050	1.080478e-13
`unboundDim1R7`	(8,7,7); 78	(1,1,0); 1	.058	1.080478e-13
`unboundDim1R8`	(9,8,8); 101	(1,1,0); 1	.068	1.080478e-13
`unboundDim1R9`	(10,9,9); 127	(1,1,0); 1	.076	1.080478e-13
`unboundDim1R10`	(11,11,10); 156	(1,1,0); 1	.086	1.080478e-13

Table 9: Complexity parameters for the SDPs in [44]. The feasible set of each SDP is an r-dimensional subspace intersected with the cone $\mathbb{S}^n_+$. To formulate each reduced SDP, a face was identified by solving LPs defined by diagonal approximations ($\mathcal{D}^d$). These LPs took (in total) t_{LPs} seconds to solve. For these examples, SeDuMi incorrectly returns an optimal value of one for the original problem. The optimal value returned for the reduced problem is very near the correct optimal value of zero.

Problem	$\mathcal{C}(\mathbb{W})^*$	$n; r$ Original	$n; r$ Reduced	$t_{LPs/SOCPs}$
`hinf12`	$\mathcal{DD}^d$	$(6, 6, 12); 77$	$(6, 2, 6); 23$	.022
`hinf13`	$\mathcal{SDD}^d$	$(7, 9, 14); 121$	$(1, 9, 7); 45$	.072

Table 10: Complexity parameters for SDPs from the DIMACS library. The feasible set of each SDP is an r-dimensional subspace intersected with the cone $\mathbb{S}^{n_1}_+ \times \mathbb{S}^{n_2}_+ \times \mathbb{S}^{n_3}_+$. To formulate each reduced SDP, a face was identified by solving LPs for `hinf12` and SOCPs for `hinf13`. These LPs/SOCPs are defined by $\mathcal{C}(\mathbb{W})^*$ and took (in total) $t_{LPs/SOCPs}$ seconds to solve.

made available. We also gave a post-processing procedure for dual solution recovery that applies generally to cone programs pre-processed using facial reduction. This recovery procedure always succeeds when the cone is polyhedral, but may fail otherwise, illustrating an interesting difference between linear programming and optimization over non-polyhedral cones.

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