SMASH PRODUCTS & THE J-HOMOMORPHISM

ARPON RAKSIT

2018-09-04

0. Introduction

Notation 0.1. Let X be a quasicompact quasiseparated scheme. Let $H_\star(X)$ denote the ∞-category of pointed motivic spaces over X. Let $\mathrm{Vec}^{\mathrm{emb}}(X)$ denote the category of vector bundles over X and injective linear bundle maps between such. And recall that given a vector bundle $V \to X$, we obtain a corresponding "sphere" $S^V \in H_\star(X)$.

The impetus for these notes was a desire for a formal ∞-categorical understanding of the following fact:

Proposition 0.2. There is a canonical symmetric monoidal functor

$$\mathrm{Vec}^{\mathrm{emb}}(X) \to H_\star(X)$$

extending the construction $V \mapsto S^V$, where $\mathrm{Vec}^{\mathrm{emb}}(X)$ is given the direct sum symmetric monoidal structure and $H_\star(X)$ is given the smash product symmetric monoidal structure.

Remark 0.3. From Proposition 0.2 it is fairly straightforward to construct the motivic J-homomorphism as a map of E_∞-spaces. See for example the beginning of §16.2 in [BH18].

We will give a proof of Proposition 0.2 in §3. The proof relies on a fact about smash products of spaces that we discuss in §2, and the theory of Day convolution symmetric monoidal structures, reviewed in §1.

1. Day convolution

Construction 1.1. Let $\mathcal{C}$ and $\mathcal{D}$ be symmetric monoidal ∞-categories. Let

$$\otimes_{\mathcal{C}} : \mathcal{C} \times \mathcal{C} \to \mathcal{C}, \quad \otimes_{\mathcal{D}} : \mathcal{D} \times \mathcal{D} \to \mathcal{D}$$

denote the underlying bifunctors. Consider the functor category $\mathcal{E} := \mathrm{Fun}(\mathcal{C}, \mathcal{D})$. Given functors $F, G \in \mathcal{E}$, we may form the composite functor

$$\mathcal{C} \times \mathcal{C} \xrightarrow{(F,G)} \mathcal{D} \times \mathcal{D} \xrightarrow{\otimes_{\mathcal{D}}} \mathcal{D}.$$

We denote a left Kan extension of this composite along $\otimes_{\mathcal{C}} : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ by $F \circledast G \in \mathcal{E}$ (when it exists). If this left Kan extension exists for all $F, G \in \mathcal{E}$, this construction extends in an evident way to a bifunctor $\circledast : \mathcal{E} \times \mathcal{E} \to \mathcal{E}$. We refer to this bifunctor as the *Day convolution product*.

Remark 1.2. A sufficient criterion for the Kan extensions in Construction 1.1 to exist is that $\mathcal{D}$ admit colimits of the diagrams

$$(\mathcal{C} \times \mathcal{C}) \times_{\mathcal{C}} \mathcal{C}_{/X} \to \mathcal{C} \times \mathcal{C} \xrightarrow{(F,G)} \mathcal{D} \times \mathcal{D} \xrightarrow{\otimes_{\mathcal{D}}} \mathcal{D}$$

for each $X \in \mathcal{C}$; here the first functor is the projection, and we are using the functor $\otimes_{\mathcal{C}}$ to form the fiber product on the left-hand side (so informally we may write objects of the fiber product as of the form $U \otimes_{\mathcal{C}} V \to X$). Under the assumption that this criterion holds, the pointwise formula for left Kan extension gives the following expression for the Day convolution product:

$$(\mathrm{F} \circledast \mathrm{G})(\mathrm{X}) \simeq \operatorname*{colim}_{\mathrm{U} \otimes_{\mathcal{C}} \mathrm{V} \to \mathrm{X}} \mathrm{F}(\mathrm{U}) \otimes_{\mathcal{D}} \mathrm{G}(\mathrm{V}).$$

We are interested in understanding when the Day convolution product extends to a symmetric monoidal structure on the functor category.

Terminology 1.3. In the situation of Construction 1.1, a *diagram of* $\mathcal{C}$*-Day shape in* $\mathcal{D}$ is a diagram of the shape $\mathrm{F}\colon (\prod_{\mathrm{I}} \mathcal{C}) \times_{\mathcal{C}} \mathcal{C}_{/\mathrm{X}} \to \mathcal{D}$, where:

- $\mathrm{X} \in \mathcal{C}$, I is a finite set, and the fiber product is formed using the multitensor product functor $\otimes_{\mathcal{C},\mathrm{I}}\colon \prod_{\mathrm{I}} \mathcal{C} \to \mathcal{C}$;
- F is obtained from a set of functors $\{\mathrm{F}_i\colon \mathcal{C} \to \mathcal{D}\}_{i \in \mathrm{I}}$ as the composite

$$(\textstyle\prod_{\mathrm{I}} \mathcal{C}) \times_{\mathcal{C}} \mathcal{C}_{/\mathrm{X}} \to \prod_{\mathrm{I}} \mathcal{C} \xrightarrow{(\mathrm{F}_i)} \prod_{\mathrm{I}} \mathcal{D} \xrightarrow{\otimes_{\mathcal{D},\mathrm{I}}} \mathcal{D},$$

  where the first functor is the projection.

We accordingly refer to colimits of diagrams of $\mathcal{C}$-Day shape as *colimits of* $\mathcal{C}$*-Day shape*.

Proposition 1.4. In the situation of Construction 1.1, assume that $\mathcal{D}$ admits colimits of $\mathcal{C}$-Day shape and that the bifunctor $\otimes_{\mathcal{D}}\colon \mathcal{D} \times \mathcal{D} \to \mathcal{D}$ preserves colimits of $\mathcal{C}$-Day shape separately in each variable. Then there is a canonical symmetric monoidal structure on $\mathcal{E} = \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ whose underlying bifunctor is the Day convolution product $\circledast\colon \mathcal{E} \times \mathcal{E} \to \mathcal{E}$.

Proof. See [Lur-A, §2.2.6]; the statements there have stronger hypotheses (and apply in the more general setting of $\mathcal{O}$-monoidal ∞-categories), but the proofs establish the claim written here. □

Terminology 1.5. We refer to the symmetric monoidal structure of Proposition 1.4 as the *Day convolution symmetric monoidal structure* on $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$.

Remark 1.6. Let $\mathcal{C}, \mathcal{D}, \mathcal{D}'$ be symmetric monoidal ∞-categories and let $\Phi\colon \mathcal{D} \to \mathcal{D}'$ be a symmetric monoidal functor. Assume that $\mathcal{D}$ and $\mathcal{D}'$ satisfy the hypotheses of Proposition 1.4, so that we obtain Day convolution symmetric monoidal structures on $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ and $\mathrm{Fun}(\mathcal{C}, \mathcal{D}')$. Then, if Φ preserves colimits of $\mathcal{C}$-Day shape, there is a canonical symmetric monoidal structure on the functor $\Phi_\star\colon \mathrm{Fun}(\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{D}')$ given by composition with Φ.

Proposition 1.7. Let $\mathcal{C}$ be a small symmetric monoidal ∞-category. Recall that the opposite category $\mathcal{C}^{\mathrm{op}}$ then canonically inherits a symmetric monoidal structure. By Proposition 1.4 we obtain a Day convolution symmetric monoidal structure on the presheaf ∞-category $\mathrm{PSh}(\mathcal{C}) := \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Spc})$, using the cartesian symmetric monoidal structure on Spc. This symmetric monoidal structure is characterized uniquely by the following two properties:

(a) the underlying bifunctor $\circledast\colon \mathrm{PSh}(\mathcal{C}) \times \mathrm{PSh}(\mathcal{C}) \to \mathrm{PSh}(\mathcal{C})$ preserves colimits separately in each variable;

(b) the yoneda embedding $\mathcal{C} \hookrightarrow \mathrm{PSh}(\mathcal{C})$ can be given a symmetric monoidal structure.

Proof. See [Lur-A, Cor. 4.8.1.12 and Rmk. 4.8.1.13]. □

In these notes, we will consider Day convolution symmetric monoidal structures for exactly one example of source category, $\mathcal{C} = [1]^{\mathrm{op}}$:

Notation 1.8. Let [1] denote the poset $\{0 \leqslant 1\}$. We equip this with the commutative monoid structure given by the "logical or" operation; this respects the ordering, hence endows [1] with the structure of a symmetric monoidal category.

We let $\langle 1 \rangle := [1]^{\mathrm{op}}$ and endow this with the same symmetric monoidal structure.

Remark 1.9. While there is of course a canonical isomorphism of posets $[1] \simeq [1]^{\mathrm{op}}$, we will adopt the convention of considering functors out of $[1]^{\mathrm{op}} = \langle 1 \rangle$.

Remark 1.10. We can understand explicitly diagrams of $\langle 1 \rangle$-Day shape:

(a) First of all, the category $\langle 1 \rangle^n$ is the n-dimensional cube poset (or its opposite); we shall write its objects as strings $q_1 q_2 \ldots q_n$ where each $q_i \in \langle 1 \rangle$. The multitensor product functor $\otimes_{\langle 1 \rangle} \colon \langle 1 \rangle^n \to \langle 1 \rangle$ takes a string $q_1 q_2 \ldots q_n$ to 1 if and only if some $q_i = 1$. It follows that the overcategory $\langle 1 \rangle^n \times_{\langle 1 \rangle} \langle 1 \rangle_{/\mathrm{X}}$ appearing in Terminology 1.3 can be described as follows:

- In the case $\mathrm{X} = 0 \in \langle 1 \rangle$, we have $\langle 1 \rangle^n \times_{\langle 1 \rangle} \langle 1 \rangle_{/\mathrm{X}} \simeq \langle 1 \rangle^n$ (via projection) as X is a final object in $\langle 1 \rangle$. Note that $\langle 1 \rangle^n$ also has a final object, $00 \ldots 0$, so that colimits over this diagram are given simply by evaluation there.
- In the case $\mathrm{X} = 1 \in \langle 1 \rangle$, we have $\langle 1 \rangle^n \times_{\langle 1 \rangle} \langle 1 \rangle_{/\mathrm{X}} \simeq (\langle 1 \rangle^n)^\circ$, where we define $(\langle 1 \rangle^n)^\circ := \langle 1 \rangle^n - \{00 \ldots 0\}$ to be the cube minus its final vertex. Let us note here that $\langle 1 \rangle^n$ is the cocone diagram over $(\langle 1 \rangle^n)^\circ$, so when we say a diagram of shape $\langle 1 \rangle^n$ is a colimit diagram, we mean that the final vertex is the colimit over the punctured cube.

(b) Let $\mathcal{D}$ be an ∞-category admitting finite limits, equipped with the cartesian symmetric monoidal structure. Let $\mathrm{F}_1, \ldots, \mathrm{F}_n \colon \langle 1 \rangle \to \mathcal{D}$ be a set of functors. Now consider the cubical diagram

$$\mathrm{F} \colon \langle 1 \rangle^n \xrightarrow{(\mathrm{F}_i)} \mathcal{D}^n \xrightarrow{\prod} \mathcal{D}$$

appearing in Terminology 1.3. On objects this sends $q_1 \ldots q_n \in \langle 1 \rangle^n$ to $\prod_{i=1}^n \mathrm{F}_i(q_i)$. We observe that the canonical map

$$\mathrm{F}(q_1 q_2 \ldots q_n) \to \mathrm{F}(q_1 0 \ldots 0) \times_{\mathrm{F}(00\ldots0)} \mathrm{F}(0 q_2 \ldots 0) \times_{\mathrm{F}(00\ldots0)} \cdots \times_{\mathrm{F}(00\ldots0)} \mathrm{F}(00 \ldots q_n)$$

is an equivalence for all $q_1 q_2 \ldots q_n \in \langle 1 \rangle^n$; this is easy to see using that (iterated) pullbacks commute with products.

Notation 1.11. Given an ∞-category $\mathcal{D}$, we let $\mathrm{Ar}(\mathcal{D}) := \mathrm{Fun}(\langle 1 \rangle, \mathcal{D})$. If $\mathcal{D}$ admits colimits of $\langle 1 \rangle$-Day shape and is equipped with a symmetric monoidal structure whose underlying bifunctor preserves colimits of $\langle 1 \rangle$-Day shape separately in each variable, then we regard $\mathrm{Ar}(\mathcal{D})$ as a symmetric monoidal ∞-category via the Day convolution structure.

Example 1.12. In Notation 1.11, we may take $\mathcal{D} = \mathrm{Spc}$, equipped with the cartesian symmetric monoidal structure. As binary products in Spc preserve colimits separately in each variable, we obtain the Day convolution symmetric monoidal structure on $\mathrm{Ar}(\mathrm{Spc}) = \mathrm{PSh}([1])$.

2. Smash products

Proposition 2.1 (Lurie)**.** There exists a unique symmetric monoidal structure on Spc_* with the following properties:

(a) the underlying bifunctor $\mathrm{Spc}_* \times \mathrm{Spc}_* \to \mathrm{Spc}_*$ preserves colimits separately in each variable;

(b) the unit object is equivalent to the zero-sphere $\mathrm{S}^0 \in \mathrm{Spc}_*$.

Terminology 2.2. We refer to the symmetric monoidal structure of Proposition 2.1 as the *smash product symmetric monoidal structure* on Spc_*. We denote the underlying bifunctor by $(\mathrm{X}, \mathrm{Y}) \mapsto \mathrm{X} \wedge \mathrm{Y}$.

The goal of this section is to give another perspective on the smash product symmetric monoidal structure on Spc_*, in terms of the Day convolution symmetric monoidal structure on $\mathrm{Ar}(\mathrm{Spc})$.

Notation 2.3. Given a map of spaces $f\colon \mathrm{U} \to \mathrm{X}$, we may form a pushout diagram

$$\begin{array}{ccc} \mathrm{X}_1 & \longrightarrow & * \\ {\scriptstyle f}\downarrow & & \downarrow \\ \mathrm{X}_0 & \longrightarrow & \mathrm{cofib}(f). \end{array}$$

The pushout space $\mathrm{cofib}(f)$ is the *cofiber* of the map f, and is canonically pointed by the right-hand vertical map. This construction determines a functor $\mathrm{cofib}\colon \mathrm{Ar}(\mathrm{Spc}) \to \mathrm{Spc}_*$.

The key fact about smash products we will use in §3 is that there is a symmetric monoidal structure on the functor $\mathrm{cofib}\colon \mathrm{Ar}(\mathrm{Spc}) \to \mathrm{Spc}_*$. We begin by observing the following weaker claim:

Lemma 2.4. For $f, g \in \mathrm{Ar}(\mathrm{Spc})$, there is a natural equivalence

$$\mathrm{cofib}(f \circledast g) \simeq \mathrm{cofib}(f) \wedge \mathrm{cofib}(g)$$

in Spc_*.

Proof. Both sides of the equivalence determine elements of the ∞-category

$$\mathrm{Fun}^{\mathrm{L}}(\mathrm{Ar}(\mathrm{Spc}), \mathrm{Fun}^{\mathrm{L}}(\mathrm{Ar}(\mathrm{Spc}), \mathrm{Spc}_*)),$$

where $\mathrm{Fun}^{\mathrm{L}}$ denotes colimit-preserving functors; it suffices to find an equivalence between them in this ∞-category. Using the universal property of $\mathrm{Ar}(\mathrm{Spc}) = \mathrm{PSh}([1])$ as a free cocompletion, we reduce to finding an equivalence between the restrictions of these elements along the yoneda embedding to

$$\mathrm{Fun}([1], \mathrm{Fun}([1], \mathrm{Spc}_*)) \simeq \mathrm{Fun}([1] \times [1], \mathrm{Spc}_*).$$

This is now a simple computation. □

While Lemma 2.4 is perhaps the first essential ingredient of a symmetric monoidal structure on $\mathrm{cofib}\colon \mathrm{Ar}(\mathrm{Spc}) \to \mathrm{Spc}_*$, rigorously it of course does not provide the whole structure. However, we can use Lemma 2.4 to obtain the whole structure, and in fact simultaneously characterize the smash product symmetric monoidal structure on Spc_* in these terms:

Proposition 2.5. (a) The functor $\mathrm{cofib}\colon \mathrm{Ar}(\mathrm{Spc}) \to \mathrm{Spc}_*$ is left adjoint to the fully faithful functor $\iota\colon \mathrm{Spc}_* \hookrightarrow \mathrm{Ar}(\mathrm{Spc})$ that identifies pointed spaces with arrows of spaces whose source is contractible.

(b) The localization $\mathrm{cofib} \dashv \iota$ is compatible (in the sense of [Lur-A, Dfn. 2.2.1.6]) with the Day convolution symmetric structure on $\mathrm{Ar}(\mathrm{Spc})$.

(c) The symmetric monoidal structure on Spc_* induced from that on $\mathrm{Ar}(\mathrm{Spc})$ by (b) has the properties of Proposition 2.1, hence agrees canonically with the smash product symmetric monoidal structure.

Proof. (a) This is more or less immediate from the definition of the cofiber as a pushout.

(b) By [Lur-A, Ex. 2.2.1.7], the condition to be shown is that if $\sigma\colon f \to g$ is a morphism in $\mathrm{Ar}(\mathrm{Spc})$ such that $\mathrm{cofib}(\sigma)\colon \mathrm{cofib}(f) \to \mathrm{cofib}(g)$ is an equivalence in Spc_*, then for any $h \in \mathrm{Ar}(\mathrm{Spc})$ the morphism $h \circledast f \to h \circledast g$ also becomes an equivalence after applying cofib. This follows from Lemma 2.4.

(c) Recall that the induced symmetric monoidal structure has the following properties: the underlying bifunctor on Spc_* is given by

$$(\mathrm{X}, \mathrm{Y}) \mapsto \mathrm{cofib}(\iota(\mathrm{X}) \circledast \iota(\mathrm{Y})),$$

and the unit is given by applying cofib to the unit in $\mathrm{Ar}(\mathrm{Spc})$, which we know is the image of the unit $0 \in [1]$ under yoneda. Condition (a) of Proposition 2.1 then holds since the same property holds for the Day convolution product on $\mathrm{Ar}(\mathrm{Spc})$ and cofib is a left adjoint, hence preserves colimits. And condition (b) of Proposition 2.1 holds by an easy computation. □

Corollary 2.6. There is a canonical symmetric monoidal structure on the functor $\mathrm{cofib}\colon \mathrm{Ar}(\mathrm{Spc}) \to \mathrm{Spc}_{\star}$, where the source is equipped with the Day convolution symmetric monoidal structure and the target the smash product symmetric monoidal structure.

The results Proposition 2.5 and Corollary 2.6 are about the ∞-category of spaces, but we may bootstrap to a more general result using the theory of tensor products of presentable ∞-categories:

Proposition 2.7. Let $\mathcal{X} \in \mathrm{Pr}^{\mathrm{L}}$ be a presentable ∞-category. Let $\mathcal{X}_{\star}$ denote the category of pointed objects in $\mathcal{X}$, i.e. the undercategory of a final object $*_{\mathcal{X}} \in \mathcal{X}$. Let $(-)_{+}\colon \mathcal{X} \to \mathcal{X}_{\star}$ denote a left adjoint to the forgetful functor.

Suppose that the cartesian symmetric monoidal structure on $\mathcal{X}$ is a presentable one, i.e. binary products in $\mathcal{X}$ preserve colimits separately in each variable (this holds for example when $\mathcal{X}$ is an ∞-topos). Then there exists a unique symmetric monoidal structure on $\mathcal{X}_{\star}$ with the following properties:

(a) the underlying bifunctor $\mathcal{X}_{\star} \times \mathcal{X}_{\star} \to \mathcal{X}_{\star}$ preserves colimits separately in each variable;

(b) the functor $(-)_{+}\colon \mathcal{X} \to \mathcal{X}_{\star}$ can be given a symmetric monoidal structure.

Proof. We first prove existence. By [Lur-A, Ex. 4.8.1.21], there is a canonical equivalence $\mathcal{X}_{\star} \simeq \mathcal{X} \otimes \mathrm{Spc}_{\star}$, where $\otimes$ denotes the tensor product in Pr^{L}. Note that our hypothesis implies that $\mathcal{X}$ has the structure of a commutative algebra object in Pr^{L}, and smash product gives $\mathrm{Spc}_{\star}$ such a structure as well, so we canonically obtain such a structure on their tensor product $\mathcal{X}_{\star}$. This is precisely a symmetric monoidal structure on $\mathcal{X}_{\star}$ satisfying (a). To see that it also satisfies (b), we note that there is also a canonical equivalence $\mathcal{X} \simeq \mathcal{X} \otimes \mathrm{Spc}$ (as commutative algebra objects in Pr^{L}), and the map $(-)_{+}\colon \mathcal{X} \to \mathcal{X}_{\star}$ is equivalent to the map obtained by applying the functor $\mathcal{X}\otimes -\colon \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}) \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$ to the map $(-)_{+}\colon \mathrm{Spc} \to \mathrm{Spc}_{\star}$ (which is a map of commutative algebra obejcts as it is the unit map for the commutative algebra structure on $\mathrm{Spc}_{\star}$ in Pr^{L}).

We now prove uniqueness. Let $\mathcal{Y}$ denote the ∞-category $\mathcal{X}_{\star}$ equipped with a symmetric monoidal structure satisfying (a, b). As $\mathcal{Y}$ is pointed, there is a canonical map $\mathrm{Spc}_{\star} \to \mathcal{Y}$ in $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$. Assumption (b) gives a map $\mathcal{X} \to \mathcal{Y}$ in $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$. As tensor product becomes coproduct for commutative algebras, we therefore obtain a canonical map $\Phi\colon \mathcal{X} \otimes \mathrm{Spc}_{\star} \to \mathcal{Y}$ in $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$. But on underlying ∞-categories this map gives the equivalence $\mathcal{X} \otimes \mathrm{Spc}_{\star} \simeq \mathcal{X}_{\star}$, so Φ is an equivalence. □

Notation 2.8. We refer to the symmetric monoidal structure of Proposition 2.7 also as the *smash product symmetric monoidal structure* on $\mathcal{X}_{\star}$, and denote the underlying bifunctor by $(\mathrm{X}, \mathrm{Y}) \mapsto \mathrm{X} \wedge \mathrm{Y}$.

Example 2.9. Let $\mathcal{C}$ be a small ∞-category and consider the case where $\mathcal{X} = \mathrm{PSh}(\mathcal{C})$ in Proposition 2.7. Then there is a canonical equivalence $\mathcal{X}_{\star} \simeq \mathrm{PSh}(\mathcal{C}; \mathrm{Spc}_{\star})$, and the smash product symmetric monoidal structure on $\mathcal{X}_{\star}$ is obtained pointwise from that on $\mathrm{Spc}_{\star}$.

Proposition 2.10. Let $\mathcal{X} \in \mathrm{Pr}^{\mathrm{L}}$ be as in Proposition 2.7. Then the statements of Proposition 2.5 and Corollary 2.6 go through with Spc replaced by $\mathcal{X}$.

Proof. Roughly speaking, we simply apply $\mathcal{X} \otimes -$ in Pr^{L}. Details omitted. □

3. The J-homomorphism

In this section, we apply the formalities of §§1 and 2 to prove Proposition 0.2. We will do this by breaking down the construction $[\mathrm{V} \in \mathrm{Vec}^{\mathrm{emb}}(\mathrm{X})] \mapsto [\mathrm{S}^{\mathrm{V}} \in \mathrm{H}_{\star}(\mathrm{X})]$ into several steps.

Notation 3.1. We work over a fixed scheme $\mathrm{X} \in \mathrm{Sch}$ throughout this section. Let Sm_{X} denote the category of finitely-presented smooth X-schemes. Let $\mathrm{Sm}_{\mathrm{X}}^{\mathrm{emb}}$ denote the wide subcategory of Sm_{X} where morphisms are the (zariski-)open embeddings.

We equip Sm_{X} with the nisnevich topology, and let $\mathrm{Shv}_{\mathrm{nis}}(\mathrm{Sm}_{\mathrm{X}})$ denote the ∞-category of sheaves of spaces on this site. The ∞-category of motivic spaces over X is defined to be the full subcategory $\mathrm{H}(\mathrm{X}) \hookrightarrow \mathrm{Shv}_{\mathrm{nis}}(\mathrm{Sm}_{\mathrm{X}})$ spanned by the $\mathbb{A}^1$-invariant sheaves, i.e. those $\mathcal{F} \in \mathrm{Shv}_{\mathrm{nis}}(\mathrm{Sm}_{\mathrm{X}})$ such that for all $\mathrm{Y} \in \mathrm{Sm}_{\mathrm{X}}$, the map $\mathcal{F}(\mathrm{Y}) \to \mathcal{F}(\mathrm{Y} \times \mathbb{A}^1)$ induced by projection is an equivalence. The inclusion of this subcategory admits a left adjoint that we will denote $\mathrm{L}_{\mathrm{mot}} \colon \mathrm{Shv}_{\mathrm{nis}}(\mathrm{Sm}_{\mathrm{X}}) \to \mathrm{H}(\mathrm{X})$.

We denote the composite of $\mathrm{L}_{\mathrm{mot}}$ with the sheafified yoneda embedding $\mathrm{Sm}_{\mathrm{X}} \to \mathrm{Shv}_{\mathrm{nis}}(\mathrm{Sm}_{\mathrm{X}})$ by $\mathrm{Y} \mapsto |\mathrm{Y}|$.

Notation 3.2. Let $\mathrm{V} \to \mathrm{X}$ be a vector bundle. Let $\mathrm{V}^{\circ} \in \mathrm{Sm}_{\mathrm{X}}$ denote the complement of the zero-section. We of course have a canonical open embedding $\mathrm{V}^{\circ} \hookrightarrow \mathrm{V}$, and the "sphere" $\mathrm{S}^{\mathrm{V}} \in \mathrm{H}_{\star}(\mathrm{X})$ is defined to be the cofiber of the induced map $|\mathrm{V}^{\circ}| \to |\mathrm{V}|$ in $\mathrm{H}(\mathrm{X})$.

Proposition 3.3. The category $\mathrm{Sm}_{\mathrm{X}}^{\mathrm{emb}}$ admits colimits of $\langle 1 \rangle$-Day shape, and binary products in $\mathrm{Sm}_{\mathrm{X}}^{\mathrm{emb}}$ preserve colimits of $\langle 1 \rangle$-Day shape separately in each variable. Thus, the cartesian symmetric monoidal structure on $\mathrm{Sm}_{\mathrm{X}}^{\mathrm{emb}}$ determines a Day convolution symmetric monoidal structure on $\mathrm{Ar}(\mathrm{Sm}_{\mathrm{X}}^{\mathrm{emb}})$.

Proof. By Remark 1.10, it suffices to show that we may construct colimits of diagrams $\mathrm{F}^{\circ} \colon (\langle 1 \rangle^n)^{\circ} \to \mathrm{Sm}_{\mathrm{X}}^{\mathrm{emb}}$ that extend to diagrams $\mathrm{F} \colon \langle 1 \rangle^n \to \mathrm{Sm}_{\mathrm{X}}^{\mathrm{emb}}$ satisfying the pullback property of Remark 1.10(b), and that these colimits are preserved in each variable of binary products. These colimits may be constructed by gluing the schemes $\mathrm{F}(10\ldots0), \mathrm{F}(01\ldots0), \ldots, \mathrm{F}(00\ldots1)$ along the open subschemes specified by the next layer of the cube; the pullback property ensures the necessary cocycle/compatibility conditions for gluing are satisfied. The statement about binary products is straightforward to see from this construction. Details omitted. □

Proposition 3.4. There is a canonical symmetric monoidal functor

$$\iota^{\circ} \colon \mathrm{Vec}^{\mathrm{emb}}(\mathrm{X}) \to \mathrm{Ar}(\mathrm{Sm}_{\mathrm{X}}^{\mathrm{emb}})$$

extending the construction $\mathrm{V} \mapsto (\mathrm{V}^{\circ} \hookrightarrow \mathrm{V})$ described in Notation 3.2. Here the source is equipped with the direct sum (i.e. cartesian) symmetric monoidal structure and the target the Day convolution symmetric monoidal structure of Proposition 3.3.

Proof. Functoriality is evident, and it is easy to see that the unit gets sent to the unit and that there is a natural isomorphism $\iota^{\circ}(\mathrm{V} \oplus \mathrm{W}) \simeq \iota^{\circ}(\mathrm{V}) \circledast \iota^{\circ}(\mathrm{W})$. The rest of the symmetric monoidal structure can be exhibited by hand as these are ordinary categories. Details omitted. □

Lemma 3.5. Let $\mathcal{X}$ be an ∞-category admitting colimits and finite limits. Suppose given a cubical diagram $\mathrm{F} \colon \langle 1 \rangle^n \to \mathcal{X}$ satisfying the pullback property of Remark 1.10(b). Then F is a colimit diagram if and only if the Čech nerve of the map

$$\mathrm{F}(10...0) \amalg \mathrm{F}(01...0) \amalg \cdots \amalg \mathrm{F}(00...1) \to \mathrm{F}(00...0)$$

is a colimit diagram (i.e. simplicial resolution of the target).

Proof. We apply Proposition A.2 to the map of simplicial sets

$$\langle 1 \rangle^{n-1} \amalg \cdots \amalg \langle 1 \rangle^{n-1} \to (\langle 1 \rangle^n)^{\circ}$$

given by the n faces of the punctured cube. This rewrites the cubical colimit as a geometric realization of a simplicial object. We identify this simplicial object with the Čech nerve using two facts: that colimits over cubes $\langle 1 \rangle^k$ are given by evaluation at the final object; and our hypothesis that F satisfies the pullback property. □

Proposition 3.6. The (restricted, sheafified) yoneda embedding

$$y\colon \mathrm{Sm}_{\mathrm{X}}^{\mathrm{emb}} \to \mathrm{Shv}_{\mathrm{nis}}(\mathrm{Sm}_{\mathrm{X}})$$

preserves colimits of $\langle 1 \rangle$-Day shape. As it also preserves finite products, there is a canonical symmetric monoidal structure on the induced functor on arrow categories

$$\mathrm{Ar}(y)\colon \mathrm{Ar}(\mathrm{Sm}_{\mathrm{X}}^{\mathrm{emb}}) \to \mathrm{Ar}(\mathrm{Shv}_{\mathrm{nis}}(\mathrm{Sm}_{\mathrm{X}})),$$

where the target too is equipped with the Day convolution symmetric monoidal structure induced by the cartesian symmetric monoidal structure on $\mathrm{Shv}_{\mathrm{nis}}(\mathrm{Sm}_{\mathrm{X}})$.[1]

Proof. Similar to Proposition 3.3, it suffices to consider colimit diagrams $\mathrm{F}\colon \langle 1 \rangle^n \to \mathrm{Sm}_{\mathrm{X}}^{\mathrm{emb}}$ satisfying the pullback property of Remark 1.10(b). By the construction of colimits in the proof of Proposition 3.3, the schemes $\mathrm{F}(10\ldots0), \mathrm{F}(01\ldots0), \ldots, \mathrm{F}(00\ldots1)$ form an open cover of the colimit $\mathrm{F}(00\ldots0)$. By [Lur-T, Prop. 6.3.2.20], it follows that the induced map

$$y(\mathrm{F}(10...0)) \amalg y(\mathrm{F}(01...0)) \amalg \cdots \amalg y(\mathrm{F}(00...1)) \to y(\mathrm{F}(00...0))$$

is an effective epimorphism in the ∞-topos $\mathrm{Shv}_{\mathrm{nis}}(\mathrm{Sm}_{\mathrm{X}})$, which by definition (see [Lur-T, below Cor. 6.2.3.5]) means that the Čech nerve of this map is a simplicial resolution. Finally, since y preserves finite limits, Lemma 3.5 now implies that the composite $y \circ \mathrm{F}\colon \langle 1 \rangle^n \to \mathrm{Shv}_{\mathrm{nis}}(\mathrm{Sm}_{\mathrm{X}})$ is also a colimit diagram, as desired. □

We can now deduce the main result:

Proof of Proposition 0.2. We take our functor to be the composite

$$\mathrm{Vec}^{\mathrm{emb}}(\mathrm{X}) \xrightarrow{\iota^{\circ}} \mathrm{Ar}(\mathrm{Sm}_{\mathrm{X}}^{\mathrm{emb}}) \xrightarrow{\mathrm{Ar}(y)} \mathrm{Ar}(\mathrm{Shv}_{\mathrm{nis}}(\mathrm{Sm}_{\mathrm{X}})) \xrightarrow{\mathrm{Ar}(\mathrm{L}_{\mathrm{mot}})} \mathrm{Ar}(\mathrm{H}(\mathrm{X})) \xrightarrow{\mathrm{cofib}} \mathrm{H}_{\star}(\mathrm{X}),$$

which on objects sends $\mathrm{V} \mapsto \mathrm{cofib}(|\mathrm{V}^{\circ}| \to |\mathrm{V}|) = \mathrm{S}^{\mathrm{V}}$. It's a fact that $\mathrm{L}_{\mathrm{mot}}$ preserves finite products (see for example [Hoy18, Prop. 3.15]), and it's a left adjoint so it preserves colimits, whence $\mathrm{Ar}(\mathrm{L}_{\mathrm{mot}})$ has a canonical symmetric monoidal structure by Remark 1.6. We have a symmetric monoidal structure on ι° by Proposition 3.4, one on $\mathrm{Ar}(y)$ by Proposition 3.6, and one on cofib by Proposition 2.10. We thus have a symmetric monoidal structure on the composite, as desired. □

A. Decomposing colimits

In [Lur-T, §4.2.3] there is a very general result on rewriting colimits of diagrams in ∞-categories in terms of colimits of simpler diagrams. Here we isolate one case of this, which is used above in proving Lemma 3.5.

Notation A.1. In this section, let $\mathcal{X}$ be a cocomplete ∞-category, let K be a simplicial set, and let $f\colon \mathrm{K} \to \mathcal{X}$ be a diagram.

Given a map of simplicial sets $p\colon \mathrm{L} \to \mathrm{K}$, we may consider the colimit of the restricted diagram $\mathrm{colim}_{\mathrm{L}}(f \circ p) \in \mathcal{X}$. This extends to a functor

$$\mathrm{colim}(f \circ -)\colon \mathrm{sSet}_{/\mathrm{K}} \to \mathcal{X}.$$

[1]Note that binary products in $\mathrm{Shv}_{\mathrm{nis}}(\mathrm{Sm}_{\mathrm{X}})$ preserve all colimits separately in each variable, as it is an ∞-topos.

Proposition A.2. Let $L_1, \ldots, L_n$ be simplicial subsets of K, and suppose that the induced map $L := \coprod L_i \to K$ is surjective. Let $L^{\times_K \bullet}\colon \Delta_+ \to \mathrm{sSet}_{/K}$ denote the augmented simplicial object given by the Čech nerve of the map $L \to K$. Then the composite diagram

$$\Delta_+^{\mathrm{op}} \xrightarrow{L^{\times_K \bullet}} \mathrm{sSet}_{/K} \xrightarrow{\operatorname{colim}(f \circ -)} \mathcal{X}$$

is a colimit diagram. Or, more informally, the colimit of $f\colon K \to \mathcal{X}$ is canonically equivalent to the geometric realization of the colimits of the restrictions $f\colon L \times_K \cdots \times_K L \to \mathcal{X}$.

Proof. We may restrict to the semisimplicial category $\Delta_{\mathrm{inj}}^{\mathrm{op}}$ since the inclusion $\Delta_{\mathrm{inj}}^{\mathrm{op}} \to \Delta^{\mathrm{op}}$ is cofinal. We now claim that [Lur-T, Cor. 4.2.3.10] applies in this situation. So that we may use the notation set there, let us set $\mathcal{J} := \Delta_{\mathrm{inj}}^{\mathrm{op}}$. We need to check that the conditions (1) and (2) of [Lur-T, Prop. 4.2.3.8] hold.

Suppose $\sigma \in K_*$ is a degenerate simplex. Since each $L \times_K \cdots \times_K L$ is a disjoint union of simplicial subsets of K, we have that $I'_\sigma = I_\sigma$. Thus condition (2) holds.

Suppose $\sigma \in K_*$ is a nondegenerate simplex. We claim that the category I_σ is a cofiltered poset, hence acyclic so that condition (1) holds. This follows straightforwardly from the decomposition

$$L \times_K \cdots \times_K L \simeq \coprod_{1 \leqslant i_1, \ldots, i_k \leqslant n} L_{i_1} \cap \cdots \cap L_{i_k},$$

by which the objects of I_σ may be identified with sequences $1 \leqslant i_1, \ldots, i_k \leqslant n$ such that $\sigma \in L_{i_j}$ for all $1 \leqslant j \leqslant k$. □

References

[BH18] Tom Bachmann and Marc Hoyois. *Norms in motivic homotopy theory*. May 24, 2018. arXiv: 1711.03061 [math.AG].

[Hoy18] Marc Hoyois. *The six operations in equivariant motivic homotopy theory*. Mar. 8, 2018. arXiv: 1509.02145 [math.AG].

[Lur-A] Jacob Lurie. *Higher Algebra*. Sept. 18, 2017.

[Lur-T] Jacob Lurie. *Higher Topos Theory*. Apr. 2017.